Math 214 Homework 11 **Your name**

> Let A, B be non-empty sets. If there is an injection $f : A \to B$, then we write $|A| \le |B|$.
>
> If there is an injection from A to B, but there is no bijection from A to B, we write $|A| < |B|$.

1. (2 points for each part.) Determine with proofs the following sets are finite, denumerable or uncountable:
   (a) $\{(x, y) : x, y \in \mathbf{N}, \text{ and } x + y = 10000\}$.
   (b) All the functions from $\mathbf{Z}_3$ to $\mathbf{R}$.
   (c) The set of all functions from $\mathbf{N}$ to $\{0, 1\}$.

2. (2 points for each part.) Let A_1, A_2, B_1, B_2 be non-empty sets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Suppose $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ are functions. Define $f : A_1 \cup A_2 \to B_1 \cup B_2$ by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2. \end{cases}$$

   (a) Show that f is a well-defined function.

   (b) If f_1, f_2 are injective, show that f is injective.

   (c) If f_1, f_2 are surjective, show that f is surjective.

3. (4 points) Let $A = \mathbf{N} - \{n^2 : n \in \mathbf{N}\}$. Construct a bijection from A to $\mathbf{N}$. Verify your answer.

4. Let $A = \{a_1, a_2, \ldots\}$ be a denumerable set.

   (a) (3 point) Prove that for every $n \in \mathbf{N}$, A can be partitioned into n denumerable sets.

   (b) (3 point) Prove that A can be partitioned into infinitely many denumerable sets.

5. (2 points for each part.) Suppose $A \subseteq B$. Prove or disprove the following.

   (a) If B is denumerable, then A is denumerable.

   (b) If A is denumerable, then B is denumerable.

   (c) If B is uncountable, then A is uncountable.

   (d) If A is uncountable, then B is uncountable.

6. (4 points) Prove that the set of infinite subsets of $\mathbf{N}$ is uncountable.

   [Hint: You may use the result in Homework 10, Problem 4.]

7. (4 points) Let $\mathbf{N}^n = \mathbf{N} \times \cdots \times \mathbf{N}$ (n times). Show that $|\mathbf{N}^n| = |\mathbf{N}|$ for any $n \in \mathbf{N}$.

8. (Extra 4 points) Construct an example of a set A with subsets B and C such that

$$|\mathbf{N}| < |C| < |B| < |A|.$$