1. Prove that if $A \subseteq B \subseteq C$ and $|C| \le |A|$, then $|A| = |B| = |C|$.

   Solution. Note: If $X \subseteq Y$, then $f : X \to Y$ defined by $f(x) = x$ for all $x \in X$ is an injection.

   Now, $A \subseteq B \subseteq C$. So, there are injections $f : A \to B$ and $g : B \to C$. If $|C| \le |A|$, then there is an injection $h : C \to A$. Note that $g \circ f$ is an injection from A to C, by Schroder-Berstein Theorem, $|C| = |A|$. Also, $g : B \to C$ is an injection, and $f \circ h : C \to B$ is an injection. So, $|B| = |C|$.

2. Let $A_j = (a_j, b_j)$ be open intervals in $\mathbf{R}$ for $j \in J$, where J is a non-empty set. Show that $|\mathbf{R}| = |\cup_{j\in J} A_j|$.

   Solution. Let $S = \cup_{j\in J} A_j$. Then $S \subseteq \mathbf{R}$ so that $f : S \to \mathbf{R}$ defined by $f(x) = x$ for all $x \in S$ is an injection.

   Because J is non-empty, there is at least one interval $A_j = (a_j, b_j)$ in the collection. We know that there are bijections $f : \mathbf{R} \to (-1, 1)$ defined by $f(x) = x/(1 + |x|)$ and $g : (-1, 1) \to (a_j, b_j)$ defined by $g(x) = a_j + (b_j - a_j)(x + 1)/2$. So, $g \circ f : \mathbf{R} \to (a_j, b_j)$ is a bijection, and $h : \mathbf{R} \to S$ defined by $h(x) = g \circ f(x)$ is an injection.

   By Schroder-Berstein Theorem $|\mathbf{R}| = |S|$.

3. (a) Prove that $\mathbf{R}$ can be partitioned into a denumerable collection of uncountable sets.

   (b) Prove that $\mathbf{R}$ can be partitioned into an uncountable collection of uncountable sets.

   Solution. (a) For every $n \in \mathbf{Z}$, let $A_n = (n, n + 1]$, which is uncountable. Then $\{A_n : n \in \mathbf{Z}\}$ is a denumerable collection of subsets of $\mathbf{R}$ forming a partition of $\mathbf{R}$.

   (b) Note that $|\mathbf{R}| = |(0, 1)| = |(0, 1) \times (0, 1)|$ by class note and previous homework. For every $x \in (0, 1)$, let $A_x = \{(x, b) : b \in (0, 1)\}$, which is uncountable. For example, $f_x(b) = (x, b)$ is a bijection from $(0, 1)$ to A_x.

   Then $\{A_x : x \in (0, 1)\}$ is a partitions of $(0, 1) \times (0, 1)$ consisting of an uncountable collection of subsets. Suppose $f : (0, 1) \times (0, 1) \to \mathbf{R}$. Then $f(A_x) = \{f(x, b) : b \in (0, 1)\} = B_x \subseteq \mathbf{R}$. One readily checks that $\{B_x : x \in (0, 1)\}$ is a partition of $\mathbf{R}$.

4. Suppose A is an infinite set, $a_0 \notin A$, and $\tilde{A} = \{a_0\} \cup A$.

   (a) Show that $|A| = |\tilde{A}|$ and hence $|\mathcal{P}(A)| = \mathcal{P}(\tilde{A})|$.

   (b) Show that $f : A \cap \mathcal{P}(A) \to \mathcal{P}(\tilde{A})$ is a well-defined injection.

   (c) Show that $|A \cap \mathcal{P}(A)| = |\mathcal{P}(A)|$.

   Solution. (a) Since A is infinite, there is $a_1 \in A$, $a_2 \in A - \{a_1\}$, $a_3 \in A - \{a_1, a_2\}$, $\ldots$, $a_{k+1} \in A = \{a_1, \ldots, a_k\}$ for every $k \in \mathbf{N}$. Thus, A contains a subset $A_0 = \{a_1, \ldots, a_n, \ldots, \}$. Define $g : A \to \tilde{A}$ by $g(a_n) = a_{n-1}$ for $n = 1, 2, \ldots$, and $g(x) = x$ for all other $x \in A$. Then it is routine to show that g is a bijection. (Say, by the divide and conquer lemma.)

   Now, $g : A \to \tilde{A}$ is a bijection. We can define a bijection $\phi : \mathcal{P}(A) \to \mathcal{P}(B)$ as follows. For every $X \subseteq A$, let $\phi(X) = Y$, where $Y = g(X) = \{g(x) : x \in X\}$. Then for every

$X \in \mathcal{P}(A)$, we have $\phi(X) = Y \in \mathcal{P}(B)$. If $\phi(X_1) = \phi(X_2) \in \mathcal{P}(B)$, then $X_1 = X_2$. Otherwise, there is $x \in X_1 - X_2$ so that $g(x) \in g(X_1) - g(X_2)$, or there is $x \in X_2 - X_1$ so that $g(x) \in g(X_2) - g(X_1)$. Hence, ϕ is one-one. Since f is bijective, $g^{-1} : B \to A$ exists. For any $Y \in \mathcal{P}(B)$, let $X = g^{-1}(Y) = \{g^{-1}(y) : y \in Y\}$. Then $\phi(X) = Y$ because $f(X) = \{g \circ g^{-1}(y), y \in Y\} = Y$. So, ϕ is onto. We get the result.

(b) Suppose f is defined as in the question. Then for every $x \in A \cup A$, if $x = a \in A$, then $f(x) = \{a_0, a\} \in \mathcal{P}(\tilde{A})$; if $x \in \mathcal{P}(A)$, then $x \in \mathcal{P}(A) \subseteq \mathcal{P}(\tilde{A})$.

Now, suppose $x_1 \neq x_2$. If $x_1, x_2 \in A$, then $f(x_1) = \{a_0, x_1\} \neq \{a_0, x_2\} = f(x_2)$. If $x_1, x_2 \in \mathcal{P}(A)$, then $f(x_1) = x_1 \neq x_2 = f(x_2) = x_2$. If $x_1 \in A, x_2 \in \mathcal{P}(A)$, then $f(x_1) = \{a_0, x_1\} \neq x_2$ as $x_2 \in \mathcal{P}(A)$ does not contain a_0. If $x_2 \in A, x_1 \in \mathcal{P}(A)$, then $f(x_2) = \{a_0, x_2\} \neq x_1$ as $x_1 \in \mathcal{P}(A)$ does not contain a_0.

Hence, f a well-defined injection.

(c) We have an injection $f : A \cup \mathcal{P}(A) \to \mathcal{P}(\tilde{A})$ and a bijection $h : \mathcal{P}(\tilde{A}) \to \mathcal{P}(A)$ by part (a). So, $h \circ f : A \cup \mathcal{P}(A) \to \mathcal{P}(A)$ is an injection. By the theorem proved in class, we see that $|A \cup \mathcal{P}(A)| = |\mathcal{P}(A)|$.