Math 214 Homework 12 (Extra credits) **Your name**

Four points for each question unless specified otherwise.

1. Prove that if $A \subseteq B \subseteq C$ and $|C| \leq |A|$, then $|A| = |B| = |C|$.

   [Hint: You have to argue that if there is an injection from C to A, then there is a bijection between A and B, and a bijection between B and C.

2. Let $A_j = (a_j, b_j)$ be open intervals in $\mathbf{R}$ for $j \in J$, where J is a non-empty set. Show that $|\mathbf{R}| = | \cup_{j \in J} A_j|$.

3. (a) Show that $\mathbf{R}$ can be partitioned into a denumerable number of uncountable sets.

   (b) Show that $\mathbf{R}$ can be partitioned into an uncountable number of uncountable sets.

   [Hint: $|\mathbf{R}| = |(0,1)| = |(0,1) \times (0,1)|$.]

4. Suppose A is an infinite set, $a_0 \notin A$, and $\tilde{A} = A \cup \{a_0\}$.

   (a) Show that $|A| = |\tilde{A}|$, and hence $|\mathcal{P}(A)| = |\mathcal{P}(\tilde{A})|$.

   (b) Show that $f : A \cup \mathcal{P}(A) \to \mathcal{P}(\tilde{A})$ defined by

   $$f(x) = \{a_0, x\} \text{ if } x \in A \text{ and } f(B) = B \text{ if } B \in \mathcal{P}(A),$$

   is a well-defined injection.

   (c) Show that $|A \cup \mathcal{P}(A)| = |\mathcal{P}(A)|$.