Let A be a set

Note: $X \in \mathcal{P}(A)$ = set of subsets of A

means X is a subset of A.

$X \subseteq \mathcal{P}(A)$

means X consists of some subsets of A

Example

The set of even numbers $= E$. , $E \in \mathcal{P}(\mathbb{N})$ / $E \subseteq \mathbb{N}$.

$\{1,2\}$; , $\mathbb{N}$, ϕ, $\{1\} \in \mathcal{P}(\mathbb{N})$.

$\{1\} \notin \mathbb{N}$.

$1 \in \mathbb{N}$

$1 \notin \mathcal{P}(\mathbb{N})$

Elements of power set of A are subsets of A

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$\mathcal{P}(\mathbb{N}) = \{\phi, \{1\}, \{2\}, \{3\}, \ldots \quad \ldots$
$\{1,2\}, \{1,3\}, \{1,4\}, \quad \ldots$
$\{2,3\} \quad \ldots \}$

$\mathcal{P}(\mathcal{P}(\mathbb{N}))$ is the set of subsets of $\mathcal{P}(\mathbb{N})$

1.4 Indexed collections of sets

We may consider a family of sets A_j with j lying in an index set J.

Then we can consider their union, intersections, etc.

Example Let $A_r = [0, r]$ with $r > 0$; $B_r = \{0, r\}$.

One may determine $\cap_{r \in R} A_r, \cup_{r \in R} A_r$, say, with $R = \{1, 2, 3\}$.

$A_r = [0, r]$ with $r > 0$

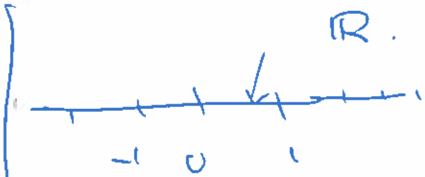
Recall : For real numbers $a < b$,

$[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$

$[a,b) = \{x \in \mathbb{R} : a \le x < b\}$

$(a,b] = \{x \in \mathbb{R} : a < x \le b\}$.

$(a,b) = \{x \in \mathbb{R} : a < x < b\}$



$A_1 = [0,1] = \{x \in \mathbb{R} : 0 \le x \le 1\}$

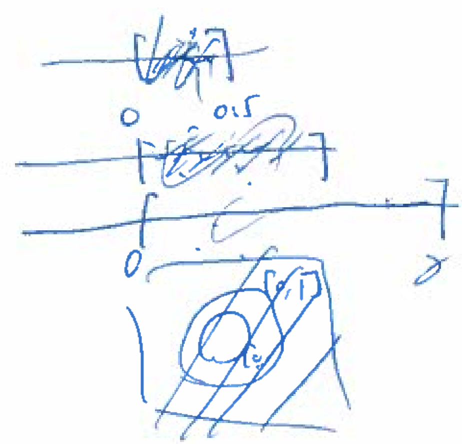
$A_{0.5} = [0, 0.5] = \{x \in \mathbb{R} : 0 \le x \le 0.5\}$

$A_{0.5} \cup A_1 = [0, 0.5] \cup [0,1]$

$= \{x : \underline{0 \le x \le 0.5}$ or $\underline{0 \le x \le 1}\}$

$= \{x : 0 \le x \le 1\}$

$= [0,1]$



$A_{0.5} \cup A_1 \cup A_2 = [0, 0.5] \cup [0,1] \cup [0,2] = [0,2]$

$A_{0.5} \cap A_1 = [0, 0.5] \cap [0,1] = [0, 0.5]$

$A_{0.5} \cap A_1 \cap A_2 = [0, 0.5] \cap [0,1] \cap [0,2] = [0, 0.5]$.

Example

Let $\boxed{B_r = [0, r) \qquad r > 0.}$

$$B_1 \cup B_\pi = [0,1) \cup [0,\pi) = [0,\pi)$$

$$B_1 \cap B_\pi = [0,1) \cap [0,\pi) = [0,1)$$

Let $J = \{1, 3, \pi, e\}$

$$B_1 \cup B_3 \cup B_\pi \cup B_e = [0,1) \cup [0,3) \cup [0,\pi) \cup [0,e) = [0,\pi)$$

$$B_1 \cap B_3 \cap B_\pi \cap B_e = [0,1)$$

Let $\Lambda = \{1, 3, \pi, e\}$.

$$\bigcup_{r \in \Lambda} B_r = B_1 \cup B_3 \cup B_\pi \cup B_e.$$

$$\bigcap_{r \in \Lambda} B_r = B_1 \cap B_3 \cap B_\pi \cap B_e$$

$$\bigcup_{r \in \{1,7,5,9,11\}} B_r = B_1 \cup B_7 \cup B_5 \cup B_9 \cup B_{11} = [0, 11) = B_{11}$$

$$\bigcup_{r \in (1,2)} B_r = \bigcup_{1<r<2} B_r = \{x \in \mathbb{R} : x \in B_r \text{ for some } r \in (1,2)\}$$

$$= \cancel{[0,2)} \{x : 0 \le x < 2\} = [0, 2) = B_2$$

$$= \bigcup_{r \in (1,2)} B_r$$

$B_{1.99}$

$1.5 \in B_{1.51}$

$1.99999999 \in B_{1.9999\ldots}$

$\cancel{B_1 \cup B_2}$ $B_1 \cup B_{...} \cdots$

Example

Let $B_r = [0, r)$ $r > 0$

$$\bigcup_{r \in (1,2) \cup (4,5]} B_r$$

$= \{ x \in \mathbb{R} : x \in B_r$ *for some* r *such that* $r \in (1,2) \cup (4,5] \}$

Let $C_r = \{0, r\}$ $r > 0 \}$

$$\bigcup_{r \in (1,2) \cup (4,5]} C_r = \{ x \in \mathbb{R} : x \in C_r \text{ for some } r \in (1,2) \cup (4,5] \}$$

$$= \{0\} \cup (1, 2) \cup (4, 5].$$

$B_1 = \{0, 1\}$

$B_{1.\ldots} = \{0, 1.\ldots\}$

$B_{1.5} = \{0, 1.5\}$

~~$B_2 = \{0, 2\}$~~

~~$B_4 = \{0, 4\}$~~

$B_{4.}$

$B_5 = \{0, 5\}$

$$\bigcap_{r \in (1,2) \cup (4,5)} C_r = \{0\}$$

Example

$A = \{x \in \mathbb{R} : x > 0\}$

$B = \{x \in \mathbb{R} : x < 0\}$

$A \cap B = \emptyset$.

Recall the notation of intervals of real numbers.

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$[a, b) =$

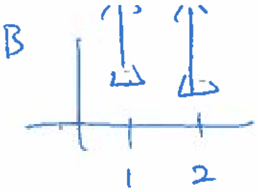
$(a, b] =$

$(a, b) =$

Example $A = [1, 5], B = [2, 6), C = \{1, 2, 3, 4, 5, 6\}$.

Example

$A = \{1, 2\}, \quad B = [1, 2)$



$$A \times B = \{(1, y) : y \in [1, 2)\} \cup \{(2, y) : y \in [1, 2)\} \subseteq \mathbb{R} \times \mathbb{R}$$

corresponds to the two line ~~sg~~ segments

L_1 with endpoints $(1,1)$ & $(1,2)$

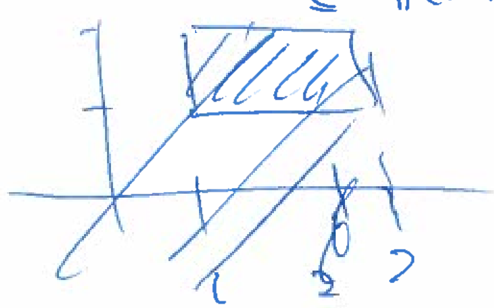
L_2 & ~~the~~ with endpoints $(2,1)$ & $(2,2)$

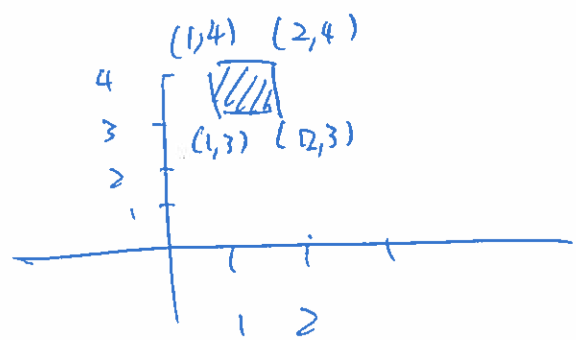
included   not included

Example

$A = [1, 2] \qquad B = [3, 4]$.

$$A \times B = \{(x, y) : 1 \le x \le 2 \,;\, 3 \le y \le 4\} \subseteq \mathbb{R} \times \mathbb{R}$$





Extension: ~~Let~~ Suppose A, B, C are sets.

$$A \times B \times C = \{(a, b, c) : a \in A,\ b \in B,\ c \in C\}$$

Remarks ① If $|A| = m$, $|B| = n$, $|C| = p$.
then $|A \times B \times C| = m \cdot n \cdot p$

② If one of A, B or C is ~~empty~~ $\emptyset$ then $A \times B \times C = \emptyset$

1.5 Partition of sets

non-empty

A partition of a set X is a collection of pairwise disjoint **nonempty** subsets whose union is X.

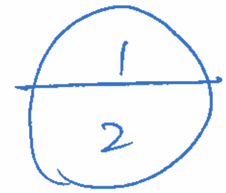
Examples $X = \{1, 2\}$ and $Y = \{\phi, \{\phi\}\}$.

Ordered pairs

1.6 Cartesian products of sets

Definition Let A, B be sets. Their Cartesian product is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$



Examples $\mathbb{N} \times \mathbb{R}$ and $[1, 2] \times [3, 4]$.

Example: $X = \{1,2\}$. $Q_1 = \{\{1\}, \{2\}\}$ is a partition

$Q_2 = \{\{1,2\}\}$ is a partition

$\{\{1\}\ \{1,2\}\}$ is NOT a partition

Example $X = \{1, 2, 3, 4\}$

$Q_1 = \{\{1,2,3,4\}\}$ — ①

$Q_2 = \{\{1\}, \{2,3,4\}\}$, $Q_3 = \{\{2\}, \{1,3,4\}\}$ Q_4 Q_5 ④

$Q_6 = \{\{1,2\}, \{3\}, \{4\}\}$ $Q_7 = \{\{1,3\}, \{2\}, \{4\}\}$ — — $\binom{4}{2} = 6$

Q_8 $\boxed{\{\{1,2\},\{3,4\}\}}$ $\boxed{\{\{1,3\},\{2,4\}\}}$ $\{\{1,4\},\{2,3\}\}$ $\quad {}_4C_2 = C(4,2)$ $\quad 3$

Q_9 $\{\{1\},\{2\},\{3\},\{4\}\}$

§1.6. Cartesian product of sets.

A, B are sets.

$$A \times B = \{ (a,b) : a \in A,\ b \in B \}$$

Example. $A = \{\text{cat}, \text{dog}, \text{pig}\}$

$B = \{ \phi, \{\phi\} \}$

$$A \times B = \{ (\text{cat}, \phi), (\text{cat}, \{\phi\}), (\text{dog}, \phi), (\text{dog}, \{\phi\}), (\text{pig}, \phi), (\text{pig}, \{\phi\}) \}$$

Note 1 $B \times A = \{ (\phi, \text{cat}), (\{\phi\}, \text{cat}), \ldots \}$

Note 2 $(1,2) \in \mathbb{R} \times \mathbb{R}$ is an ordered pair.

$(1,2) = \{ x \in \mathbb{R} : 1 < x < 2 \}$ is ~~the~~ an open interval

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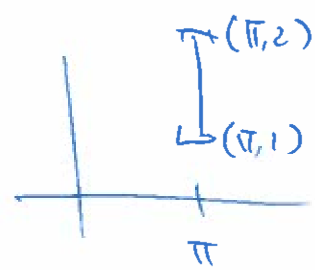
Example $A = \{\pi\}$ $B = [1,2]$



$A \times B \subseteq \mathbb{R} \times \mathbb{R}$

$A \times B = \{ (\pi, y) : y \in [1,2] \}$

is the line segment in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

joining the endpoints $(\pi,1)$ & $(\pi,2)$.

Exaple

$A = \{\phi, \boxed{\{\phi\}}\}$

$Q_1 = \{\boxed{\{\phi\}}, \boxed{\{\{\phi\}\}}\}$

$Q_2 = \{\{\phi, \{\phi\}\}\}$

Compare

$A = \{1, 2\}$

$\{\{1\}, \{2\}\}$

$\{1, 2\}$