Recall: Quiz 1 will take place on Thurs.

Q1. From HW1. Q1 $\rightarrow$ Q6 (not X-credit problems)

Q2. Truth table (easy).

Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 2 Logic

We study the mathematical language to read and write correct (logical) mathematics.

2.1 Statements

- A **statement** is a sentence/assertion which we can decide that it is true (T) or false (F).
- Examples. The integer 57 is a prime number. It is raining now. $2 + 4 = 6$.

  F F T
- An **open sentence** is an assertion with one or more variables chosen from a domain S.
- Example. $P(x)$: $x > 3$. Here the domain can be $\mathbb{N}$, $\mathbb{Q}$, or $\mathbb{R}$.
- A statement may be true (T) or false (F); two statements have 4 possible combination;
- 3 statements have 8 possible combination; ...; n statements have 2^n combination.
- We may draw the truth table for that.

Example

How are you?

Have you had breakfast?

} not statements

I have breakfast. T

Example

Let $S = \{$ ~~boy~~ People in 214 $\}$.

$P(x)$: x is a boy.

$x = CK$, ~~P(x)~~ "CK is a boy" is a true ~~statement~~. T

If $x = CK$, then $P(x)$ is true T

$x = Liz$ "Liz is a boy" F

$x = Liz$ "$P(x)$" F

Example

$S = \mathbb{N}$. $P(x): x > 3$.

Truth set of $P(x)$ with domain $S = \mathbb{N}$ is

$$T = \{x \in \mathbb{N} : P(x) \text{ is true}\}$$
$$= \{x \in \mathbb{N} : x > 3\}$$
$$= \{4, 5, 6, 7, 8, \ldots\}$$

$\{3, 2, 1\}$

$\|$

Remark: We can [illegible] $P(x)$ is false$\}$

2.2/2.3 Negation, disjunction, and conjunction of statements

- Negation of P, denoted by $\sim P$; "It is not P."
- Disjunction: P or Q, denoted by $P \vee Q$;
- Conjunction: P and Q, denoted by $P \wedge Q$.
- Examples and truth tables.

Example P: It is raing

Q: 2+4=6

Let P be a statement. Then we have the following truth table of the negation statement of P. "denial"

P	$\sim P$
T	F
F	T

Let P, Q be statements. We can form the disjunction & conjunction of the two statements

P and Q wedge

Conjunction

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$(2+4=6) \wedge (5 \text{ is prime})$

$(2+4=6) \wedge (5 \text{ is not a prime})$

Let ~~P, Q~~

P or Q

Disjunction

P	Q	$P \vee Q$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

2.4/2.5/2.6 Implication and biconditional

- Implication: If P then Q (also, P implies Q), denoted by $P \Rightarrow Q$.
- The statement P is the hypothesis/premise, and the statement Q is the conclusion.
- Biconditional: P is equivalent to Q (also, P if and only if Q), denoted by $P \iff Q$.
- Examples and truth tables.

Conditional

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	✓ T
T	F	F	T
F	T	T	✓ F
F	F	T	T

Example: If "~~you~~ one gets 93% or above in Coursework" then "the person will get an A for the Course".

If Q then P ; Q implies P

Q	P	$Q \Rightarrow P$
T	T	T
T	F	F
F	T	T
F	F	T

If P is true, Q is ~~true~~ false: (Assume)

$P \wedge Q$	F
$P \vee Q$	T
$P \Rightarrow Q$	F
~~$P \Leftarrow Q$~~	
$Q \Rightarrow P$	T

Biconditional

P if and only if Q.

"one gets an A in this class" if and only if "one gets 93% or above in Coursework"

P	Q	$P \iff Q$
T	T	T
T	F	F
F	T	F
F	F	T

"# Logical equivalence"

~~Biconditional and~~

~~$P \Rightarrow$~~

The following compound statements have the same truth table.

~~P~~	~~Q~~	~~$P \Rightarrow Q$~~	~~$Q \Rightarrow$~~	~~$Q \Rightarrow P$~~
~~T~~	~~T~~			
~~T~~	~~F~~			
~~F~~	~~T~~			
~~F~~	~~F~~			

P	Q	$P \Leftrightarrow Q$	$\overset{R}{P \Rightarrow Q}$	$\overset{S}{Q \Rightarrow P}$	$\overbrace{(P \Rightarrow Q)}^{r} \wedge \overbrace{(Q \Rightarrow P)}^{s}$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

We say that

"$P \Leftrightarrow Q$" is logically equivalent to

"$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$".

2.8/2.9 Logical equivalence and properties

Two compound statements R and S are logically equivalent, denoted by $R \equiv S$ when they have the same truth values for different combination of the component statements.

Examples. $\equiv (\sim P) \vee Q$

$(P \Rightarrow Q) \equiv$ ~~$(\sim P) \vee Q$;~~

P	Q	$P \Rightarrow Q$	$\sim P$	$(\sim P) \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Commutative, associative, and distributive laws for $\vee$ and $\wedge$.

$\equiv$

De Morgan's Law.

"$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$" $\boxed{\text{Associative}}$

"$P \wedge Q$" $\equiv$ "$Q \wedge P$" $\boxed{\text{Commutative}}$

P	Q	R	$P \wedge Q$	$(P \wedge Q) \wedge R$	$Q \wedge R$	$P \wedge (Q \wedge R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

$\equiv$

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$\equiv$

Similarly, $P \vee Q \equiv Q \vee P$

$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

Distributive

$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

$a(b+c) = a \cdot b + a \cdot c$.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

$\equiv$

De Morgan's Law.

$$\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$$

$$\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$$

P	Q	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P$	$\sim Q$	$\sim P \vee \sim Q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Similarly, we can prove $\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

Application to set theory:

Let A, B be subsets of the universal set U.

$$A \cap B = \{x \in U : (x \in A) \text{ and } (x \in B)\}.$$

$$\overline{A \cap B} = \{x \in U : \sim[(x \in A) \text{ and } (x \in B)]\}$$

$$\overline{A} = \{x \in U : \sim(x \in A)\}$$

$$\overline{B} = \{x \in U : \sim(x \in B)\}$$

$$\overline{A} \cup \overline{B} = \{x \in U : [\sim(x \in A)] \text{ or } [\sim(x \in B)]\}$$

So
$$\overline{A \cap B} = \{x \in U : \sim[\underbrace{(x \in A)}_{P} \text{ and } \underbrace{(x \in B)}_{Q}]\}$$
$$= \{x \in U : [\sim(x \in A)] \text{ or } [\sim(x \in B)]\}$$
$$= \overline{A} \cup \overline{B}.$$

$$\therefore \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Similarly, $\overline{A \cup B} = \overline{A} \cap \overline{B}$