Math 214 Foundations of Higher Mathematics Quiz 1 Name: Sample Solution

1. (12 points) (a) Give an example of three sets A, B and C each with **2 elements** such that

$$A \neq B \text{ and } A - C = B - C.$$

$A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{2, 3\}$

Then $A - C = \{1\}$, $B - C = \{1\}$

(b) Let $R = \{1, \{1\}\}$. Determine $\mathcal{P}(R)$ and $\mathcal{P}(R) - R$, where $\mathcal{P}(R)$ is the power set of R.

$\mathcal{P}(R) = \{\phi, R, \{1\}, \{\{1\}\}\}$

~~P(R) − R) = { φ, R, {{1}} }~~

$\mathcal{P}(R) - R = \{\phi, R, \{\{1\}\}\}$

2. (8 points) Show that "$P \Rightarrow Q$" and "$(\sim Q \Rightarrow (\sim P)$" are logically equivalent, i.e., the statements have the same truth tables.

P	Q	$P \Rightarrow Q$	$\sim Q$	$\sim P$	$(\sim Q) \Rightarrow (\sim P)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

$\equiv$

2.7 Tautologies and contradiction

Tautology In a compound statement, all possible combination of the components yield T.

Contradiction In a compound statement, all possible combination of the components yield F.

Example

"$P \vee \sim P$" is a Tautology

P	$\sim P$	$P \vee (\sim P)$
T	F	T
F	T	T

every case is true.

"$P \wedge \sim P$" is a contradiction

P	$\sim P$	$P \wedge (\sim P)$
T	F	F
F	T	F

everything is F

Example

~~$(P \wedge Q) \Leftrightarrow V$~~ "$(P \Rightarrow Q) \vee (\sim Q)$"

P	Q	$P \Rightarrow Q$	$\sim Q$	$(P \Rightarrow Q) \vee (\sim Q)$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

is a Tautology

Remark: ~~If~~ A compound statement "$S(P, Q, R, \ldots)$" is a tautology, ~~then~~ "$\sim S(P, Q, R, \ldots)$" is a contradiction.

if and only if

2.10 Quantifiers

There exists: $\exists$; for all: $\forall$.

These usually go with open statements with variables from a domain.

Examples For every real number r, $(r+1)^2 > 0$. $\forall\, r \in \mathbb{R}, (r+1)^2 > 0$.

There is a real number r such that $r^2 = -1$. $\exists\, r \in \mathbb{R}, r^2 = -1$.

For an open statement $P(x)$, $x \in S$.

is not a statement until you put a special value x_0 in $P(x)$.

Then we can decide $P(x_0)$ is true or false.

Example: $P(x): x-3=4$. $x \in \mathbb{R}$.

Then $P(7)$ is true

$P(x_0)$ is false whenever $x \neq 7$.

$T_1 = \{x \in \mathbb{R} : \text{"}P(x)\text{" is true}\} = \{7\}$

$Q(x): x \geq 0$, $x \in \mathbb{R}$

$T_2 = \{x \in \mathbb{R} : \text{"}Q(x)\text{" is true}\}$

$= [x, \infty) = \{x \in \mathbb{R} : x \geq 0\}$.

$T = \{x \in \mathbb{R} : \text{"}P(x) \Rightarrow Q(x)\text{" is true}\}$.

$\therefore x \in T$.

Hence $T = \mathbb{R}$.

Example:

$x=0$ $P(0) \Rightarrow Q(0)$
$F \Rightarrow T$. / T $\therefore 0 \in T$

$x=1$ $P(1) \Rightarrow Q(1)$
$F \Rightarrow T$ / T $\therefore 1 \in T$

$x=7$ $P(7) \Rightarrow Q(7)$
$T \Rightarrow T$ / T $\therefore 7 \in T$.

Case 1 For $x = 7$, ~~then~~ $P(7) \Rightarrow Q(7)$ is true. So $7 \in T$.

Case 2 For $x \neq 7$, $P(x)$ is false, then $P(x) \Rightarrow Q(x)$ is true regardless

Example:

There is $x \in \mathbb{R}$ such that $x \geq 0$

$\{\exists x \in \mathbb{R}\}, x \geq 0$ / $\exists x \in S, P(x)$

is a true statement

Reason. $1 \in \mathbb{R}, \ 1 \geq 0$

Example For all (every / any) real number x, $\cancel{x \in \mathbb{R}}$ $x \geq 0$ / $\forall x \in S, P(x)$

is a false statement

because $-1 \in \mathbb{R}$, $-1 \not\geq 0$.

Example For all $x \in \mathbb{R}$, $x^2 \geq 0$.

is a true statement

Proof (Explanation: Case 1 For $x \geq 0$,

$x^2 \geq 0$

Case 2 For $x < 0$

we have $-x > 0$.

$x^2 = (-x)(-x) > 0$,

Example. It is not true that (There is $x \in \mathbb{R}$, $x \geq 0$) $\sim(\exists x \in S, P(x))$

$\equiv$ For every $x \in \mathbb{R}$, $x < 0$ $\equiv (\forall x \in S) \sim P(x)$

Basic Patterns:

$\sim(\forall x \in S, P(x))$	$\sim(\exists x \in S)(P(x))$
$\equiv (\exists x \in S), \sim P(x)$.	$\equiv (\forall x \in S) \sim P(x)$

General Patterns:

1.
$$\sim(\forall x \in S_1, \forall y \in S_2, P(x,y))$$
$$\equiv (\exists x \in S_1) \sim(\forall y \in S_2, P(x,y))$$
$$\equiv (\exists x \in S_1)(\exists y \in S_2) \sim P(x,y).$$

2.
$$\sim(\exists x \in S_1, \exists y \in S_2, P(x,y))$$
$$\equiv (\forall x \in S_1) \sim(\exists y \in S_2, P(x,y))$$
$$\equiv \forall x \in S_1\ \forall y \in S_2, \sim P(x,y).$$

3.
$$\sim(\exists x \in S_1, \forall y \in S_2, P(x,y))$$
$$\equiv (\forall x \in S_1) \sim(\forall y \in S_2, P(x,y))$$
$$\equiv (\forall x \in S_1\ \exists y \in S_2, \sim P(x,y))$$

4.
$$\sim(\forall x \in S_1, \exists y \in S_2, P(x,y))$$
$$\equiv (\exists x \in S_1) \sim(\exists y \in S_2, P(x,y))$$
$$\equiv \exists x \in S_1\ \forall y \in S_2 \sim P(x,y).$$