Basic Patterns:

$\sim(\forall x\in S, P(x))$	$\sim(\exists x\in S)(P(x))$
$\equiv (\exists x\in S), \sim P(x)$.	$\equiv (\forall x\in S) \sim P(x)$

General Patterns:

① $\sim\left(\forall x\in S_1, \left[\forall y\in S_2,\ P(x,y)\right]\right)$

$\equiv (\exists x\in S_1) \sim(\forall y\in S_2, P(x,y))$

$\equiv (\exists x\in S_1)(\exists y\in S_2) \sim P(x,y)$.

② $\sim(\exists x\in S_1, \exists y\in S_2,\ P(x,y))$

$\equiv (\forall x\in S_1) \sim(\exists y\in S_2, P(x,y))$

$\equiv \forall x\in S_1\ \forall y\in S_2, \sim P(x,y)$

③ $\sim(\exists x\in S_1, \forall y\in S_2, P(x,y))$

$\equiv (\forall x\in S_1) \sim(\forall y\in S_2, P(x,y))$

$\equiv (\forall x\in S_1\ \exists y\in S_2, \sim P(x,y))$

④ $\sim(\forall x\in S_1, \exists y\in S_2, P(x,y))$

$\equiv (\exists x\in S_1) \sim(\exists y\in S_2, P(x,y))$

$\equiv \exists x\in S_1\ \forall y\in S_2 \sim P(x,y)$.

Example from HW 2

#5 $\sim$ "For every integer $n>0$, there is some real numbers $x>0$ such that $x<\frac{1}{n}$"

$\sim \left(\forall n\in S_1, \exists x\in S_2, P(x,n)\right)$

$\equiv \exists n\in S_1 \; \forall x\in S_2 \; \sim P(x,n)$

$S_1 = \mathbb{N}$

$S_2 = \{x\in\mathbb{R}: x>0\}$

$P(x,n): \; x<\frac{1}{n}$

There is positive integer n such that for all positive real numbers x we have

$$x \geq \frac{1}{n}$$

Try

$n=1$, $x = \boxed{\frac{1}{2}} < \frac{1}{1}$

$n=10$, $x = \boxed{\frac{1}{20}} < \frac{1}{10}$

$n=100$, $x = \boxed{\frac{1}{200}} < \frac{1}{100}$

2. For every $n\in\mathbb{N}$, let $x=\frac{1}{2n}>0$ & $x=\frac{1}{2n}<\frac{1}{n}$ because $n<2n$

#6. $\alpha>0$, $S_\alpha = (-\alpha,\alpha)$

(a) $\sim\left(\forall\alpha\in(0,1) \; \exists\beta\in(0,1) \text{ s.t. } S_\alpha\subset S_\beta\right)$

$\equiv \left(\exists\alpha\in(0,1) \; \forall\beta\in(0,1) \text{ s.t. } S_\alpha\not\subset S_\beta\right)$

$\forall\alpha\in(0,1) \; \exists\beta\in(0,1) \; S_\alpha\subset S_\beta$

$-\beta \quad -0.1 \quad 0.1 \quad \beta \quad 1$

$\alpha=0.1$ ~~$\beta\in(0,1)$~~ $\beta=0.2$

$\alpha=0.9$ $\beta=0.99$

$\alpha=0.999$ $\beta=0.9999$

$\alpha \quad 1$

$\beta = \frac{1+\alpha}{2}$

(b) $\exists\alpha\in(0,1)\,\forall\beta\in(0,1), S_\alpha\subset S_\beta$

Try $\alpha=0.1$ $-0.1 \quad 0.1$ X $\because \beta=0.05$

$-(0.05, 0.05)$

$\alpha=0.000000001$ X

$\beta=\frac{\alpha}{2}$

$\alpha>0 \Rightarrow \alpha<2\alpha$

$\Rightarrow \frac{\alpha}{2}<\alpha$

Math 214 Foundations of Higher Mathematics C.K. Li

Chapters 3 – 5: Proofs

We practice how to read and write mathematical proofs.

General discussions from Chapters 3 and 5; and examples from Chapter 4.

$x \in S$

$P(x)$	$Q(x)$	$P(x) \Rightarrow Q(x)$
T	T	T
F	T	T
F	F	T

Terminology Axiom, Theorem, Lemma, Corollary, etc.

In mathematics, we often try to establish "$P(x) \Rightarrow Q(x)$", for $x \in S$.

We also consider "$P(x)$ if and only if $Q(x)$".

§3.1 and §3.2 Trivial proof Assume $x \in S$ and $P(x)$ is true. Then verify Q(x).

We ignore trivial statements/situations: (a) when $P(x)$ is always false, (b) when $Q(x)$ is always true.

In such cases, the proof is **trivial/vacuous**.

Examples If x is real such that $x < 0$, then $x^2 + 1 > 0$. T.

If $x \in \mathbf{R}$ such that $x^2 - 2x + 3 < 0$, then $20^3 > 100$. T

If 3 is even, then $n \in \mathbf{Z}$ is divisible by 5. T

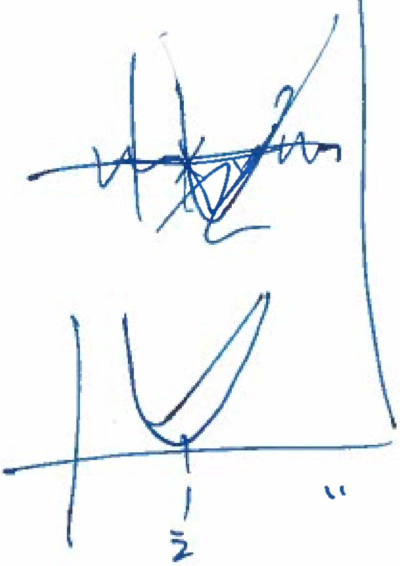
Prove or disprove: If $r \in (0, 1)$, then $\boxed{1/4 \geq r(1 - r).}$ T →

Prove or disprove: If $r \in (0, 1)$, then $1/4 > r(1 - r)$.

$Q(x)$ is always true

$x \in (0,1)$

$Q(x)$: $\frac{1}{4} \geq r(r-r)$.



To prove: " $1/4 \geq r(1-r)$ "

i.e. " $1 \geq 4r(1-r)$ "

i.e., $1 \geq 4r - 4r^2$

i.e., $4r^2 - 4r + 1 \geq 0$

i.e., $(2r-1)^2 \geq 0$ ~~except~~ is always true.

which is always true.

$\therefore$ If $r \in (0,1)$ then $\frac{1}{4} \geq r(1-r)$.

" If $r \in (0,1)$ then $\frac{1}{4} > r(1-r)$." is a false statement.

Because $r = \frac{1}{2} \in (0,1)$ and

$\frac{1}{4} \not> \frac{1}{2}(1-\frac{1}{2})$.

§3.2 **Direct proof** We assume P(x) and verify Q(x) (to get the "theorem").

Examples If $n \in \mathbf{Z}$ and $5n - 7$ is odd, then $9n + 2$ is even. [An integer m is even (odd) if ....]

2 If a, b, c are integers such that a divides b, and b divides c, then a divides c.

[For integers x and y, we say that x divides y if $y = xz$ for some $z \in \mathbf{N}$.]

3 If a is not divisible by 3 and b is not divisible by 3, then $a^2 - b^2$ is divisible by 3.

Let A and B be sets. Then $A \cup B = A \cap B$ if and only if $A = B$.

Let A, B, C be sets such that C is non-empty. Then $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

What if C is empty?

Note: $n \in \mathbb{Z}$ is even if $n = 2k$, $k \in \mathbb{Z}$

$n \in \mathbb{Z}$ is odd if $n = 2k-1$ $k \in \mathbb{Z}$

Assume $n \in \mathbb{Z}$, $5n-7$ is odd.

i.e., $5n - 7 = 2k - 1$.

$5n = 2k + 6$

$9n+2 = 5n + 4n + 2 = 2k + 6 + 4n + 2$

$= 2(k + 3 + 2n + 1)$.

$9n + 2 = 2m$ $\quad m = k+4+2n \in \mathbb{Z}$

$\therefore$ $9n+2$ is even.

Alternatively

Fact: If x, y are integers, y is odd $x-y$ is odd, then x is even.

Proof Let $x, y \in \mathbb{Z}$. y is odd $x - y$ is odd.

i.e. $y = 2k-1$, $x - y = 2m - 1$, $k, m \in \mathbb{Z}$

Then $x = (x-y) + y = (2m-1) + (2k-1)$

$= 2(m+k-1) = 2l$

$\therefore$ x is even $\quad l = m+k-1 \in \mathbb{Z}$

If $n \in \mathbb{Z}$, $x = 5n-7$ is odd,

Consider $y = 9n+2$. Then $y - x = 9n + 2 - (5n-7)$

$= 4n + 9 = 2\underbrace{(2n+5)}_{2n+5} - 1$

is odd.

(2) If $a, b, c \in \mathbb{Z}$ $a|b$ and $b|c$, then $a|c$.

Note: $x, y \in \mathbb{Z}$, $x|y$ means $y = xq$, $q \in \mathbb{Z}$ ✓

Assume $a, b, c \in \mathbb{Z}$, $a|b$, $b|c$.

i.e. $b = aq_1$, $c = bq_2$, $q_1, q_2 \in \mathbb{Z}$

$\therefore$ $c = bq_2 = a(q_1 q_2)$

✓ $c = aq$ with $q = q_1 q_2 \in \mathbb{Z}$.

$\therefore$ $a|c$

Assume $a, b, c \in \mathbb{Z}$.

(3) a is not divisible by 3 and b is not divisible by 3.

$a = 3q_1 + r_1$, $r_1 = 1$ or 2, $b = 3q_2 + r_2$ $r_2 = 1, 2$.

Case 1° $r_1 = r_2 = 1$ $a^2 - b^2 = (3q_1 + 1)^2 - (3q_2 + 1)^2 = (9q_1^2 + 6q_1 + 1) - (9q_2^2 + 6q_2 + 1)$

$= 3(3q_1^2 + 2q_1 - 3q_2^2 - 2q_2)$

$= 3m$, $m = 3q_1^2 + 2q_1 - 3q_2^2 - 2q_2 \in \mathbb{Z}$

$\cancel{a^2 - b^2 = 3m}$ $\cancel{\text{for } m = \quad \in \mathbb{Z}}$.

$\therefore$ $a^2 - b^2$ is divisible by 3

Case 2° $r_1 = 2$, $r_2 = 1$ $a^2 - b^2 = (3q_1 + 2)^2 - (3q_2 + 1)^2$

$= (9q_1^2 + 12q_1 + 4) - (9q_2^2 + 6q_2 + 1)$

$= 3(3q_1^2 + 4q_1 - 3q_2^2 - 2q_2 + 1)$

$= 3m$ with $m = \underbrace{\quad\quad}_{} \in \mathbb{Z}$.

$\therefore$ $a^2 - b^2$ is divisible by 3

Case 3° $r_1 = 1$, $r_2 = 2$. Similar to Case 2°

Case 4° $r_1 = 2$, $r_2 = 2$. Similar to Case 1°.