more examples

Prove: Let A, B be sets. Then $A \cup B = A \cap B$ if and only if $A = B$

Proof. $(\Rightarrow)$ Assume $A \cup B = A \cap B$.

Then $\underline{A} \subseteq \underline{A \cup B} = \underline{A \cap B} \subseteq B$

Also $B \subseteq A \cup B = A \cap B \subseteq A$.

$\therefore A = B$

$(\Leftarrow)$ Assume $A = B$.

Then $A \cup \underline{B} = A \cup \underline{A} = A$

$A \cap B = A \cap A = A$

$\therefore A \cup B = A \cap B$

Example A, B are sets. $C \neq \phi$.

Prove $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

Have to show $(\Rightarrow)$, $(\Leftarrow)$.

$(\Rightarrow)$: Assume $A \times C \subseteq B \times C$.

If $(x,y) \in A \times C$ then $(x,y) \in B \times C$.

~~Need to show if~~ $a \in A$, let $c \in C$, $C \neq \phi$.

so that $(a,c) \in A \times C$. $\therefore$ $(a,c) \in B \times C$

$\therefore$ $a \in B$, $c \in C$.

Then $a \in B$.

$\therefore$ $A \subseteq B$.

$A = \{1,2,3\}$
$B = \{1,2,3,4\}$
$C = \{5\}$.

$A = \{1,2,3\}$
$B = \{1,2,3\}$
$C = \{5\}$

$(\Leftarrow)$ Assume $A \subseteq B$.

~~Then~~ If $(x, y) \in A \times C$.

then $x \in A \wedge y \in C$

So $x \in B \wedge y \in C$ by assumption

$\therefore$ x,

$\therefore$ ~~then~~ $(x,y) \in B \times C$

$\therefore$ $A \times C \subseteq B \times C$.

Note If $A \subseteq B$, then $A \times C \subseteq B \times C$.

But $A \times C \subseteq B \times C \not\Rightarrow A \subseteq B$ if $C = \phi$

Example. $A = \{1,2\}$, $C = \phi$

$B = \{1\}$

But $A \times C = \phi$

$B \times C = \phi$

$A \times C = B \times C$

$A \times C \subseteq B \times C$

§3.3. Proof by contrapositive To prove "If $P(x)$ then $Q(x)$." we prove "If $\sim Q(x)$, then $\sim P(x)$."

Examples ~~If $n \in \mathbf{Z}$ is such that $15n$ is even, then $9n$ is even.~~

$\boxed{\text{An integer } n \text{ is odd (even) if and only if } n^2 \text{ is odd (even).}}$

$(P \Rightarrow Q) \equiv ((\sim Q) \Rightarrow (\sim P))$

Proof

Let $n \in \mathbb{Z}$. Prove

n is odd if and only if n^2 is odd.

($\Rightarrow$) Assume n is odd, i.e., $n = 2l+1$, $l \in \mathbb{Z}$

Then $n^2 = (2l+1)^2 = 4l^2 + 4l + 1 = 2(2l^2+2l)+1$

$\therefore n^2 = 2m+1$ with $m = \boxed{2l^2+2l} \in \mathbb{Z}$.

$\therefore n^2$ is odd.

($\Leftarrow$) Assume n^2 is odd. Need to prove n is odd.

Assume, $n^2 = 2m+1, m \in \mathbb{Z}$ Need to prove $n = 2l+1$, $l \in \mathbb{Z}$

We prove by using the contrapositive of the statement:

Assume $\sim(\cancel{n \text{ is odd}})$ n is even

$n = 2l$, $l \in \mathbb{Z}$

$\therefore n^2 = 4l^2 = 2(m)$ $m = 2l^2 \in \mathbb{Z}$

$\therefore \sim(\cancel{n^2 \text{ is odd}})$. n^2 is even

§5.2 Proof by contradiction To prove $P \Rightarrow Q$, show that $P \wedge \sim Q$ is impossible. $\sim(P \Rightarrow Q) \equiv \sim(P \vee \sim Q)$

Also, to prove P, assume $\sim P$ and derive a contradiction. $\equiv P \wedge \sim Q$.

Examples The sum of a rational number and an irrational number is irrational.

The number $\sqrt{2}$ is irrational. (If $x = \sqrt{2}$, then x is irrational.)

There are infinitely many prime numbers.

(If S is the set of primes, then S has infinitely many elements.)

If $x, y \in \mathbf{R}$ are positive, then $\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$.

R	~R
T	F
F	T

If ~~Assume~~ $x = \sqrt{2} \in \mathbb{R}$, then x is irrational.

Assume $x = \sqrt{2}$.

$x = \sqrt{2} \neq \frac{m}{n}$, $m, n \in \mathbb{Z}$.

$\therefore$ x is irrational

Indirect proof: Assume $x = \frac{m}{n}$ is rational

then $x \neq \sqrt{2}$

$P \wedge$

Proof by contradiction Assume ~~If~~ $x = \sqrt{2}$,

& Assume $\boxed{\sqrt{2} = \frac{m}{n}, \ m, n \in \mathbb{Z}.}$ (*)

P	Q	P⇒Q
T	T	T

P	Q	P⇒Q
T	F	F

We may assume m, n have no common factors.

Then $2 = \frac{m^2}{n^2}$ $\therefore$ $2n^2 = m^2$

cannot happen

So m^2 is even and so is m. Thus $m = 2k$ and $2n^2 = m^2 = (2k)^2 = 4k^2$.

$\therefore$ $n^2 = 2k^2$ is even and so is n.

Thus m & n have a common

$\therefore$ There is a contradiction !!!

Example:

Pg There are infinitely many prime numbers.

If S is the set of prime numbers, then S is infinite.

$n \in \mathbb{N}$, $n \geq 2$ is a prime

if $n = ab$, $1 < a, b < n$, $a, b \in \mathbb{N}$

Proof:

Assume S is the set of prime.

and assume the contrary that

S is finite.

So $S = \{p_1, p_2, \cdots, p_n\}$.

Consider $\boxed{q = p_1 p_2 \cdots p_n + 1}$.

Case 1° q is a prime number.

then $q \notin S$. which is a contradiction.

Case 2° q is not a prime number

then q has a prime factor p.

and $p \neq p_i$ for any i.

(because p_i is not a factor of q.

as $q = p_i \boxed{m_i} + 1$)

So p is a prime number not in S,

which is a contradiction.

Example The sum of a rational number and an irrational number is irrational.

Proof: Reformulate:

If $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$, $n \neq 0$

and $y \neq \frac{a}{b}$ for any $a, b \in \mathbb{Z}$, $b \neq 0$.

then $x + y \neq \frac{r}{s}$ for any $r, s \in \mathbb{Z}$ $s \neq 0$

Proof: Assume $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$. $n \neq 0$

Assume $y \neq \frac{a}{b}$ for any $a, b \in \mathbb{Z}$, $b \neq 0$.

Assume $x + y = \frac{r}{s}$ for any $r, s \in \mathbb{Z}$, $s \neq 0$

Then $y = (x+y) - x = \frac{r}{s} - \frac{m}{n}$

$$= \frac{rn - ms}{sn} = \frac{p}{q} \qquad p = rn - ms \in \mathbb{Z}, \quad q = sn \in \mathbb{Z}, \quad sn \neq 0$$

$\therefore$ $y \in \mathbb{Q}$

$\therefore$ a contradiction

Example: If $x, y \in \mathbb{R}$, $x, y > 0$, then $\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$.

Proof: Assume $x, y \in \mathbb{R}$, $x, y > 0$.

Assume the contrary that $\sqrt{x} + \sqrt{y} = \sqrt{x+y}$.

Then $(\sqrt{x} + \sqrt{y})^2 = (\sqrt{x+y})^2$

$\therefore$ $x + 2\sqrt{xy} + y = x + y$

$\therefore$ $2\sqrt{xy} = 0$ So $x = 0$ or $y = 0$, a contradiction

Hint on homework . . :

\# 8 .

$$S = \{ n \in \mathbb{N} : \sqrt{n} \text{ is irrational} \}$$

$$A = \{ 2m^2 : m \in \mathbb{N} \} \subseteq S .$$

Reason: $x \in A$, $x = 2m^2$

So that $\sqrt{x} = \sqrt{2}\, m$ is irrational

To prove :

S has no maximum number.

Assume $N \in S$ is max.

i.e, If $x \in S$ then $x \leq N$.

Hint. Then consider $2N^2$.