Sample Solution

Math 214 QUIZ 2 Name: ____________________

1. Let $m \in \mathbf{N}$. Show that if m^2 is divisible by 5, then m is divisible by 5.

We prove the contrapositive: Suppose m is not divisible by 5. Prove that $5 \nmid m^2$

Proof Assume $5 \nmid m$. Consider 4 cases.

- $m = 5q+1$, $m^2 = 25q^2 + 10q + 1 = 5(5q^2 + 2q) + 1$,
- $m = 5q+2$, $m^2 = 25q^2 + 10q + 4 = 5(5q^2 + 2q) + 4$,
- $m = 5q+3$, $m^2 = 25q^2 + 10q + 4 = 5(5q^2 + 2q + 1) + 4$,
- $m = 5q+4$, $m^2 = 25q^2 + 10q + 16 = 5(5q^2 + 2q + 3) + 1$.

In all cases, m^2 is not divisible by 5.

2. Prove that $\sqrt{5}$ is irrational.

Hint: Suppose $\sqrt{5} = p/q$ so that p, q have no common factor. Then $5q^2 = p^2$ and use the result in Problem 1 to conclude that p is divisible by 5. Then argue that q is also divisible by 5....

Proof Suppose, (by contradiction) that $\sqrt{5} = \frac{p}{q}$, $p, q \in \mathbb{N}$ have no common factors.

Then $5q^2 = p^2$. So $5 \mid p$ by Problem 1.

So $p = 5k$, $k \in \mathbb{Z}$, and $5q^2 = (5k)^2 = 25k^2$.

Thus $q^2 = 5k^2$. and $5 \mid q$ by Problem 1.

Hence p, q have 5 as a common factor,

which is a contradiction !!! which contradicts the fact that p & q have no common factor.

So $\sqrt{5}$ is irrational.

Shield

Spear 矛盾

(e) Prove/disprove "There exist odd integers a & b such that $4 \mid (3a^2+7b^2)$.

(a,b)	$3a^2+7b^2$
$(1,1)$	10
$(1,3)$	$3+63=66$
$(1,5)$	$3+7\times25=$

$3+7(24+1)$
$= 7\cdot24+3+7$
$= 7\cdot24+8+2$

Let
$a = 2k+1$
$b = 2l+1$ be odd

Then $3a^2+7b^2$
$= 3(4k^2+4k+1) + 7(4l^2+4l+1)$
$= 4(3k^2+3k+7l^2+7l) + 3+7$
$= 4(3k^2+3k+7l^2+7l) + 8+2$
$= 4(3k^2+3k+7l^2+7l+2) + 2$
$\therefore\ 3a^2+7b^2 = 4n+2.$
Cannot be a multiple of 4.

$\therefore$ So the original statement is false.

$\sim[\nexists\ a, b$ odd integers, ~~$4 \mid (3a^2+7b^2)$~~ $]$

~~i.e., the original sta~~

The negation of the original statement.
$\sim[\exists$ ~~a b~~ odd integers a, b, such that $4 \mid (3a^2+7b^2)]$
$\equiv \forall$ odd integers a, b, $4 \nmid (3a^2+7b^2)$.
is a true statement

(d) There is a real number x such that

$$x^6 + x^4 + 1 = 2x^2$$

i.e.,

$$-x^6 = x^4 - 2x^2 + 1$$

$$= (x^2-1)^2 .$$

$-x^6 \leq 0$, $(x^2-1)^2 \geq 0$.

$\therefore$ the equality holds only when $\underline{-x^6 = 0} = (x^2-1)^2$,

i.e., $x = 0$ & $x^2 - 1 = 0$, which is impossible

$\therefore$ no real number x satisfies the equation.

$$\underbrace{x^6 + x^4 + 1 - 2x^2}_{f(x)} = 0$$

In other words, the original statement is false.

Examples

(a) Suppose $x, y \in \mathbf{R}$. Then $\frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy$.

(b) Let $S_a = [0, a]$ for $a > 0$. Determine (with explanation/proof) the following:

$$\bigcup_{a\in[1,2]} S_a, \quad \bigcup_{a\in(1,2)} S_a, \quad \bigcap_{a\in[1,2]} S_a, \quad \bigcap_{a\in(1,2)} S_a.$$

(a) Prove. Consider

$$\left(\tfrac{1}{3}x^2 + \tfrac{3}{4}y^2\right) - (xy)$$
$$= \tfrac{1}{12}\left[4x^2 + 9y^2 - 12xy\right]$$
$$= \tfrac{1}{12}(2x - 3y)^2$$
$$\geq 0$$

$\therefore \quad \frac{1}{3}x^2 + \frac{3}{4}y^2 \geq xy$

To prove $E_1 \geq E_2$ (a,b,c,d)

Consider $E_1 - E_2$

$=$

$=$

$=$

≥ 0 ✓

$\therefore E_1 \geq E_2$

(b) $\alpha > 0$, $S_\alpha = [0, \alpha]$.

Example: $\alpha = 0.1$, $S_{0.1} = [0, 0.1]$

$\alpha = \pi$, $S_\pi = [0, \pi]$

$\alpha = 10$, $S_{10} = [0, 10]$.

Proof ~~Determine~~ Determine

Claim: $\bigcup_{\alpha\in[1,2]} S_\alpha = [0, 2]$

To prove $X = Y$ we need to prove $(X \subseteq Y)$: $(Y \subseteq X)$:

$(\subseteq)$: $\bigcup_{\alpha\in[1,2]} S_\alpha \subseteq [0,2]$

Let $x \in \bigcup_{\alpha\in[1,2]} S_\alpha$.

i.e., there is $\alpha_0 \in [1,2]$ s.t. $x \in S_{\alpha_0} = [0, \alpha_0]$

$\therefore x \in [0,2]$ as ...

$(\supseteq)$ $[0,2] \subseteq \bigcup_{\alpha\in[1,2]} S_\alpha$

Let $x \in [0,2]$.

Then $x \in S_2$.

$\therefore x \in \bigcup_{\alpha\in[1,2]} S_\alpha$

$\mathbb{R}$

$S_1 = [0, 1]$

$S_{1.5} = [0, 1.5]$

$S_2 = [0, 2]$

(b.2) Claim: $\bigcup_{\alpha \in (1,2)} S_\alpha = [0,2)$.

Proof:

$(\subseteq)$ Let $x \in \bigcup_{\alpha \in (1,2)} S_\alpha$

There is $\alpha_0 \in (1,2)$

such that $x \in S_{\alpha_0} = [0, \alpha_0]$

$\subseteq$

$\therefore\ x \in [0,2)$ because $\alpha_0 < 2$

$\alpha_0 < \frac{\alpha_0 + 2}{2} < 2$

$(\supseteq)$ Let $x \in [0,2)$

~~$x \in [0,2$~~

~~Then~~ Let $\alpha_0 = \frac{2+x}{2} < 2$.

and $x < \alpha_0$.

Then $x \in [0, \alpha_0] = S_{\alpha_0}$

$\therefore\ x \in \bigcup_{\alpha \in (1,2)} S_\alpha$

$x \quad 2$

$\frac{x+2}{2} = \alpha_0$