① To prove "$P \iff Q$".

(1.a) "$P \Rightarrow Q$"

(1.b) "$Q \Rightarrow P$"

② Contrapositive of "$P \Rightarrow Q$".

"$\sim Q \Rightarrow \sim P$"

③ You must read ~~[illegible]~~ and understand the examples in the textbook.

④ Come to my office, homework sessions.

Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 6 Mathematical Induction

We develop a machinery to show that the open statement $P(n)$ is true for all natural numbers (or all natural numbers starting from n_0).

6.1 The principle of mathematical induction

A set of real number may or may not have a **least** element. → $\mathbb{R}$, $[0,1)$. does not satisfy WOP

Theorem If the least number exits in a set of real number, then it is unique.

The well ordering principle of natural numbers Every non-empty subset of **N** has a least element. Peano's axiom.

Remarks The well ordering principle fails for **Q** or **R**.

One can prove the well ordering principle for other subsets of **Z**. For example, the set of integers larger than −1000, the set of nonnegative even integers. Same proof as (4).

Example (1) $T=\{n\in\mathbb{N}: n \text{ is even}\}$. has a smallest number 2.

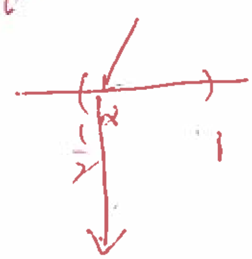
(2) $\mathbb{Z}$ does not satisfy WOP because ~~for any~~ the subset $S=\mathbb{Z}\subseteq\mathbb{Z}$ has no smallest element. because for any $r\in S$, $r-1\in S$, and $r-1<r$.

(3) $T=[0,1)$

$S=\{\cancel{2}1\}\not\subseteq T$.

$S=(\frac{1}{2},1)\subseteq T$ has no minimum. because for any $\alpha\in(\frac{1}{2},1)$

because $\dfrac{\alpha+\frac{1}{2}}{2}<\alpha$



$\dfrac{\alpha+\frac{1}{2}}{2}\in(\frac{1}{2},1)$ ∴ T does not satisfy WOP.

$$\boxed{\frac{1}{2}<\frac{\alpha+\frac{1}{2}}{2}<\alpha}$$

Theorem: If ~~A~~ $S \subseteq \mathbb{R}$ and S has a minimum, then it is unique: i.e., there cannot be two minima for a subset S of $\mathbb{R}$.

Proof: Recall x is a minimum of S if

$$x \le y \quad \text{for every } y \in S.$$

So Prove the result by contradiction.

Suppose $x = \min S$, i.e., $x \le y$ for all y in S.

$\hat{x} = \min S$. $\hat{x} \le y$ for all y in S.

$y = \hat{x}$ is in S. $\therefore$ $x \le y = \hat{x}$.

$z = x$ is in S $\therefore$ $\hat{x} \le z = x$.

$\therefore$ $x \le \hat{x}$ and ~~$\hat{x} \le x$~~. $\hat{x} \le x$

$\therefore$ $x = \hat{x}$.

(4) $T = \{ r \in \mathbb{Z} : r > -5 \}$

T satisfies WOP

Proof Suppose $S \subseteq T$, $S \neq \phi$.

Case 1° $S \subseteq \mathbb{N}$.

Then S has a minimum by WOP of $\mathbb{N}$.

Case 2° If $S \cap \{-4, -3, -2, -1, 0\} \neq \phi$

Then min of S will be

the minimum of $S \cap \{-4, -3, -2, -1, 0\}$.

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(5) $T = \{ q \in \mathbb{Q} : q \geqslant 0 \}$ does not satisfy WOP

P Let $S = \{ q \in \mathbb{Q} : q > 0 \} \subseteq T$

not ~~exempt~~ empty.

For any $x \in S$, $x = \frac{m}{n}$, $m, n \in \mathbb{N}$.

we have $\frac{x}{2} = \frac{m}{2n} \in S$ and

$\frac{x}{2} < x$. $\therefore S$ has no minimum.

The principle of mathematical induction Let $P(n)$ be an open statement with $n \in \mathbf{N}$. Suppose we can established the following two statements.

(a) $P(1)$ is true. (b) If $P(k)$ is true then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{N}$.

Remark This is the domino effect!

Proof. Show that $S = \{n \in \mathbf{N} : P(n) \text{ is false}\}$ no smallest element. So, it is empty.

Proof of the principle of M.I.:

Assume (a) $P(1)$ is true, (b) $P(k)$ is true $\Rightarrow P(k+1)$ is true for $k \geq 1$.

Want to show: $T = \{n \in \mathbb{N} : P(n) \text{ is true}\} = \mathbb{N}$.

Consider $F = \{n \in \mathbb{N} : P(n) \text{ is false}\}$

Want to show $F = \phi$.

Assume by contradiction, $F \neq \phi$.

Note: $F \subseteq \mathbb{N}$. $\therefore$ $\exists$ a minimum element $m \in F$.

By (a) ~~Because~~, $P(1)$ is true. So $m > 1$.

So $m-1 \in \mathbb{N}$ and must be in T.

But then $P(m-1)$ is true will ensure $P(m)$ is true by (b), $\therefore$ $m \in T$

which ~~is a contradiction~~ contradicts $m \in F$.

Examples (a) $P(n)$: $1 + \cdots + n = n(n+1)/2$, $n \in \mathbb{N}$

(b) $P(n)$: $1^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.

$P(1)$: $1 = \frac{1\cdot 2\cdot 3}{6} = 1$ ✓

$P(2)$: $1^2+2^2 = \frac{2\cdot 3\cdot 5}{6} = 5$

$P(3)$: $1^2+2^2+3^2 = \frac{3\cdot 4\cdot 7}{6} = 14$

$\vdots$

$P(1)$: $1 = \frac{1\cdot 2}{2} = 1$ ✓

$P(2)$: $1+2 = \frac{2\cdot 3}{2} = 3$

$P(3)$: $1+2+3 = \frac{3\cdot 4}{2} = 6$

$P(4)$: $1+2+3+4 = \frac{4\cdot 5}{2} = 10$.

$\vdots$

(a) Prove $P(n)$: $1 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Proof. By induction:

(1) $P(1)$: $1 = \frac{1\cdot 2}{2}$ which is true

(2) Assume $P(k)$ is true, $k \geq 1$.

i.e., $1 + \cdots + k = \frac{k(k+1)}{2}$

Then $\underbrace{1 + \cdots + k}_{} + (k+1)$

$= \frac{k(k+1)}{2} + (k+1)$ by induction assumption

$= \cancel{\frac{(k+1)(k+2)}{2}} = (k+1)\left[\frac{k}{2}+1\right] = (k+1)\left(\frac{k}{2}+\frac{2}{2}\right) = \frac{(k+1)(k+2)}{2}$

$\therefore$ $P(k+1)$ is true

By Principle of M.I. $P(n)$ holds for all $n \in \mathbb{N}$.

(b) Prove $P(n)$: $1^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

(1) $P(1)$: $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$, is true.

(2) Assume $P(k)$ is true. $k \geq 1$.

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Then $\underbrace{1^2 + 2^2 + \cdots + k^2}_{} + (k+1)^2$ ← L.H.S of $P(k+1)$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by induction assumption)}$$

$$= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)] = \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$\therefore$ $P(k+1)$ is true ← R.H.S. of $P(k+1)$.

By Principle of MI $P(n)$ holds $\forall n \in \mathbb{N}$.