$$\sum_{\ell=1}^{n} \frac{1}{(\ell+1)(\ell+2)} = \frac{n}{2n+4}$$

$P(1)$ : $\ell=0$ $\ell=1$ $\frac{1}{2\cdot 3} = \frac{1}{2\cdot 1+4}$

$P(2)$ : $\ell=0$ $\ell=1$, $\ell=2$ $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{2}{2\times 2+4}$

$P(3)$ : $\ell=1$ $\ell=2$ $\ell=3$ $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{3}{2\times 3+4}$

**Examples** (c) $P(n) : \frac{1}{2\cdot 3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$. , $n \in \mathbb{N}$

(d) $P(n) : 4|(5^n - 1)$.

(e) $P(n) : 6|(n^3 - n)$.

[We can prove (e) by the method of minimum counter example. See Section 6.3.]

(c)

① $P(1)$ : $\frac{1}{2\cdot 3} = \frac{1}{2\cdot 2+4}$ is true. ✓

② Assume $P(k)$ is true, $\boxed{k \geq 1}$

i.e., $\frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2k+4}$.

Consider $P(k+1)$.

$$\left[\frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)}\right] + \frac{1}{(k+2)(k+3)}$$

$$= \frac{k}{2k+4} + \frac{1}{(k+2)(k+3)}$$ by induction assumption

$$= \frac{1}{(k+2)}\left[\frac{k}{2} + \frac{1}{(k+3)}\right] = \frac{k+1}{2(k+3)} = \frac{k+1}{2(k+1)+4}$$

$$\frac{1}{(k+2)} \cdot \left[\frac{k(k+3)+2}{2(k+3)}\right]$$

$$= \frac{1}{(k+2)} \frac{[k^2+3k+2]}{2(k+3)} = \frac{1}{k+2} \frac{(k+1)(k+2)}{2(k+3)}$$

$\therefore P(k+1)$ is true.

By the principle of Mathematical Induction

$P(n)$ is true for all $n \in \mathbb{N}$.

$P(1)$ : $\frac{1}{2\cdot 3} = \frac{1}{2\cdot 1+4}$

$P(2)$ : $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{2}{2\cdot 2+4}$

$P(3)$ : $\frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{3}{2\times 3+4}$

(d) $P(n)$ : $4 \mid (5^n - 1)$, $n \in \mathbb{N}$

Prove $P(n)$ holds for all $n \in \mathbb{N}$ by induction.

Step 1. $P(1)$ : $5^1 - 1 = 4$ is divisible by 4. $\therefore P(1)$ holds

Step 2. Assume $P(k)$ is true. i.e., $\boxed{5^k - 1 = 4q}$ for some $q \in \mathbb{Z}$

Consider $P(k+1)$

$$5^{k+1} - 1 = 5^k \cdot 5 - 1 = 5^k \cdot 5 - 1\cdot 5 + 1\cdot 5 - 1$$

$$= (5^k - 1)\cdot 5 + 4 = 4q\cdot 5 + 4 = 4(5q+1)$$

is divisible by 4.

$\therefore$ $P(k+1)$ is true

By PMI, $P(n)$ is true for all $n \in \mathbb{N}$.

(e) Consider $P(n): 6 \mid (n^3 - n)$.

Prove by induction that $P(n)$ holds for all $n \in \mathbb{N}$.

Step 1. $P(1)$ : $1^3 - 1 = 0$ is divisible by 6.

$\therefore P(1)$ holds.

Step 2. Assume $P(k)$ holds. $k \geq 1$

i.e., $k^3 - k = 6q$ for some $q \in \mathbb{Z}$.

Consider $P(k+1)$

$$(k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - (k+1)$$

$$= k^3 + 3k^2 + 3k - k.$$

$$= (k^3 - k) + 3k(k+1).$$

$$= 6q + 3(k)(k+1)$$

$$= 6q + 3 \cdot 2 \cdot r$$ because $\frac{k(k+1)}{2} = r \in \mathbb{Z}$ as $k$ or $k+1$ is even.

$$= 6(q+r)$$

is divisible by 6.

$\therefore P(k+1)$ holds

By PMI, ~~P(n)~~ the result follows.

**Principle of MI - 3** Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$.

(b) For $k \geq m$ if $P(j)$ is true for all $j = m, \ldots, k$, then $P(k+1)$ is true.

$P(m), P(m+1), \cdots, P(k), \Rightarrow P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

**Remark** We use all the previous fallen dominoes to ensure the next domino would fall also.

**Example** Every integer $n \geq 2$ is a prime or a product of primes.

**6.2/6.4 More general principles**

**Principle of MI - 2** Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$. ✓ (b) For $k \geq m$, if $P(k)$ is true then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

**Remark** This follows from the fact that $S = \{n \in \mathbf{Z} : n \geq m\}$ is well ordered.

**Examples** (a) For every $n \geq 5$, $2^n > n^2$.

(b) Let $A_1, \ldots, A_n$ be sets with $n \geq 2$. Then $\overline{\cup_{i=1}^{n} A_i} = \cap_{i=1}^{n} \overline{A_i}$.

(c) Suppose $n \geq 0$. If a set with $n$ elements then its power set has $2^n$ elements.

(a) Prove by induction that $P(n)$: $2^n > n^2$ for $n \geq 5$. $n \in \mathbb{N}$

| $n$ | $2^n$ | $n^2$ |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 4 | 4 |
| 3 | 8 | 9 |
| 4 | 16 | 16 |
| 5 | 32 | 25 |
| 6 | 64 | 36 |
| ⋮ | ⋮ | ⋮ |

Proof.

Step 1. $P(5)$: $2^5 = 32 > 25 = 5^2$.

$\therefore P(5)$ holds.

Step 2. Assume $P(k)$ holds, $\boxed{k \geq 5}$

i.e., $\boxed{2^k > k^2}$, i.e., $2^k - k^2 > 0$

Consider $P(k+1)$

$2^{k+1} - (k+1)^2$

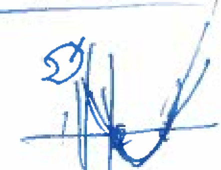
$= 2^k \cdot 2 - (k^2 + 2k + 1)$

$= (2^k - k^2) + (2^k - 2k - 1)$

$> 2^k - 2k - 1 > k^2 - 2k - 1 \geq 5k - 2k - 1 = 3k - 1 > 0$

$\geq 15 - 1 = 14 > 1$

$(k+1)^2$



$k^2 - 2k - 1 \geq k^2 - 2 \cdot 5 - 1$

$\cancel{k^2 - 7k = k(k-7)}$

**Recursive sequences** Induction is useful in proving formulas and properties of recursive sequences, i.e., sequences $\{a_1, a_2, \ldots\}$ defined by specification of $a_1, \ldots, a_k$ and a relation/formula expressing $a_m$ in terms of the previous $a_1, \ldots, a_k$, for $m > k$.

**Examples** (a) Prove that $a_n = n^2$ for all $n \in \mathbf{N}$ if the sequence $\{a_1, a_2, \ldots\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 4,\ \text{and } \boxed{a_n = 2a_{n-1} - a_{n-2} + 2} \text{ for } n \geq 3.$$

$a_m = 2a_{m-1} - a_{m-2} + 2 \quad m \geq 3$

(b) Find a formula for $a_n$ with a proof if the sequence $\{a_1, a_2, \ldots\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 2,\ \text{and } a_n = a_{n-1} + 2a_{n-2} \text{ for } n \geq 3.$$

$a_3 = 2 \cdot a_2 - a_1 + 2$
$= 2 \cdot 4 - 1 + 2$
$= 9.$

$a_4 = 2 \cdot a_3 - a_2 + 2$
$= 2 \cdot 9 - 4 + 2$
$= 16$

(a) Conjecture: $P(n): \ a_n = n^2, \ n \in \mathbb{N}$

Step 1. $P(1)$; $P(2)$

$a_1 = 1^2, \ a_2 = 2^2$ are true.

Step 2. Assume $P(1), \ldots, P(k)$ are true, $k \geq 2$

$a_1 = 1^2, \ a_2 = 2^2, \ \ldots, \ a_k = k^2$

Consider $P(k+1)$.

$a_{k+1} = 2 \cdot a_k - a_{k-1} + 2$
$= 2 \cdot k^2 - (k-1)^2 + 2$
$= 2k^2 - (k^2 - 2k + 1) + 2 = k^2 + 2k + 1$
$= (k+1)^2$

$\therefore$ $P(k+1)$ is true.

By PMI, the result follows.