Principle of MI - 3 Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$. $P(m), P(m+1), \cdots, P(k) \Rightarrow P(k+1)$ is true.

(b) For $k \geq m$ if $P(j)$ is true for all $j = m, \ldots, k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

Remark We use all the previous fallen dominoes to ensure the next domino would fall also.

Example Every integer $n \geq 2$ is a prime or a product of primes.

$P(n)$

Proof: By induction on $n \geq 2$.

$P(2)$: $n=2$ is a prime. ✓

Assume $P(2), P(3), \cdots, P(k)$ are true for $k \geq 2$.

Consider $k+1$.

Case 1° $k+1$ is a prime. Then $P(k+1)$ holds.

Case 2° $k+1$ is not a prime. So it is a composite number.

and $k+1 = ab$ such that $a, b \in \mathbb{N}$

$1 < a, b < k+1$.

By induction assumption

$P(a), P(b)$ hold.

So $a = p_1 \cdots p_m$, a product of primes

or $m=1$, i.e., a is a prime

$b = q_1 \cdots q_l$, a product of primes

or $l=1$, i.e., b is a prime.

Hence $ab = p_1 \cdots p_m q_1 \cdots q_l$ is a product of primes.

So $P(k+1)$ holds.

By PMI $P(n)$ holds $\forall n \geq 2$

6.2/6.4 More general principles

Principle of MI - 2 Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$. ✓ (b) For $k \geq m$, if $P(k)$ is true then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

Remark This follows from the fact that $S = \{n \in \mathbf{Z} : n \geq m\}$ is well ordered.

$\overline{A_1 \cup \cdots \cup A_n} = \overline{A_1} \cap \cdots \cap \overline{A_n}$

Examples (a) For every $n \geq 5$, $2^n > n^2$. ✓ ✓

(b) Let $A_1, \ldots, A_n$ be sets with $n \geq 2$. Then $\overline{\cup_{i=1}^n A_i} = \cap_{i=1}^n \overline{A_i}$.

(c) ~~Suppose $n \geq 0$.~~ If a set with n elements then its power set has 2^n elements. ✓✓

(d) Prove that $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n$ $\quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

(b) Proof by induction on $n \geq 2$.

$P(2)$: $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ holds because of the De Morgan Law.

Assume $P(k)$ holds. i.e.

$\overline{A_1 \cup \cdots \cup A_k} = \overline{A_1} \cap \cdots \cap \overline{A_k}$, $k \geq 2$.

Consider $P(k+1)$

$$\overline{A_1 \cup \cdots \cup A_{k+1}} = \overline{X \cup A_{k+1}} \quad \text{if } X = A_1 \cup \cdots \cup A_k.$$
$$= \overline{X} \cap \overline{A_{k+1}} = \left(\overline{A_1 \cup \cdots \cup A_k}\right) \cap \overline{A_{k+1}}$$
$$= \left(\overline{A_1} \cap \cdots \cap \overline{A_k}\right) \cap \overline{A_{k+1}}$$
$$= \overline{A_1} \cap \cdots \cap \overline{A_{k+1}}$$

(c) Prove that $P(n)$: A set A with n element has 2^n subsets. $n \geq 0$.

Proof by induction: $P(0)$: $A = \phi$, $\mathcal{P}(A) = \{\phi\}$.

$\therefore$ A with 0 element has $2^0 = 1$ subset ϕ.

Assume $P(k)$ holds, $k \geq 0$

i.e. If $A_k = \{a_1, \ldots, a_k\}$ has k elements.

then $\mathcal{P}(A_k) = \{S_1, \ldots, S_{2^k}\}$.

Consider $P(k+1)$. Suppose $A_{k+1} = \{a_1, \cdots, a_k, a_{k+1}\}$

Want to show that there are 2^{k+1} subsets.

Consider 2 types of subsets of A_{k+1}.

Case 1° Those subsets without a_{k+1}, i.e. subsets of $\{a_1, \dots, a_k\}$. and there are 2^k so many by induction assumption.

Case 2 Those subsets containing a_{k+1}.
They must be of the form:

$$S_1 \cup \{a_{k+1}\},\ S_2 \cup \{a_{k+1}\},\ \dots\ S_{2^k} \cup \{a_{k+1}\}.$$

& there are 2^k so many.

Combining, we have $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ subsets of A_{k+1}. $\therefore$ $P(k+1)$ holds.

By PMI. $P(n)$ holds $\forall\, n \geq 0$.

$P(n)$:

(**Binomial Theorem**) For $n \in \mathbb{N}$,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n$$

when $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ # of ways of choosing k objects out of n objects

$= C(n,k)$

Example

$$(x+y)^0 = 1$$
$$(x+y)^1 = x+y = \binom{1}{0}x^1 + \binom{1}{1}y^1$$
$$(x+y)^2 = x^2+2xy+y^2$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$
$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4.$$
$\vdots$

Pascal' Triangle

楊輝三角

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}$$

Lemma $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

for any $n > k > 0$

$$\frac{3!}{1!\,2!} = \frac{3\cdot \cancel{2\cdot 1}}{1\cdot \cancel{2\cdot 1}} = 3$$

$\binom{n}{0} = 1$

$\binom{n}{n} = 1$

~~Reason~~ Proofs ① Choose k objects out of n objects $a_1, \ldots, a_n$ can be done in two ways

(a) Choose $k-1$ objects from $a_1, \ldots, a_{n-1}$, and then add a_n to the collection.

So there ... $\binom{n-1}{k-1}$ ways.

(b) Choose k objects from $a_1, \ldots, a_{n-1}$.

So there are $\binom{n-1}{k}$ ways.

Combining $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

<u>Proof 2.</u>

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$

$$= \frac{(n-1)!}{(k-1)!\,(n-k-1)!}\left[\frac{1}{(n-k)} + \frac{1}{k}\right] = \left[\frac{n}{(n-k)\,k}\right]\frac{(n-1)!}{(k-1)!\,(n-k-1)!}$$

$$= \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

Prove $P(n)$: $(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n$.

$n \geq 1$

<u>Proof.</u> $(x+y)^1 = x+y = \binom{1}{0}x + \binom{1}{1}y$

$\therefore$ $P(1)$ holds.

Assume $P(m)$ holds, $m \geq 1$

$$(x+y)^m = \binom{m}{0}x^m + \cdots + \binom{m}{m}y^m$$

Now ~~Consider~~ $(x+y)^{m+1} = (x+y)(x+y)^m$

$$= (x+y)\left[\binom{m}{0}x^m + \binom{m}{1}x^{m-1}y + \cdots + \binom{m}{m}y^m\right]$$

$$= \left[\binom{m}{0}x^{m+1} + \binom{m}{1}x^m y + \binom{m}{2}x^{m-1}y^2 + \cdots + \binom{m}{m}xy^m\right]$$

$$+ \quad \binom{m}{0}x^m y + \binom{m}{1}x^{m-1}y^2 + \cdots + \binom{m}{m-1}xy^m + \binom{m}{m}y^{m+1}$$

$$= \binom{m+1}{0}x^{m+1} + \binom{m+1}{1}x^m y + \binom{m+1}{2}x^{m-1}y^2 + \cdots + \binom{m+1}{m}xy^m + \binom{m+1}{m+1}y^{m+1}$$

$\therefore$ $P(m+1)$ holds $\therefore$ $P(n)$ holds. Hence