$\underline{R}$ on $S$ Reflexive, Symmetric, Transitive

**Theorem** Let $R$ be an equivalence relation on a non-empty set $S$, and let $[a] = \{x \in S : (a, x) \in R\}$ be the equivalence class of $a \in S$.

(a) For $a, b \in S$, one and only one of the following holds.

(a.i) $(a, b) \in R$ and $[a] = [b]$, (a.ii) $(a, b) \notin R$ and $[a] \cap [b] = \emptyset$.

(b) The set $P = \{[a] : a \in S\}$ of equivalence classes forms a partition of $S$, i.e., $S$ is a disjoint union of the nonempty subsets $[a]$.

$[(\frac{1}{2}, \frac{1}{2})] = \{(x,y) : x^2+y^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{2}\}$

**Examples** (a) Let $S = \mathbf{Z}$, and $(a, b) \in R$ if $\boxed{a \equiv b \pmod n}$. Then the equivalence classes are $[0], [1], \ldots, [n-1]$.

(b) Let $S = \mathbf{R} \times \mathbf{R} = \mathbf{R}^2$, and $((x_1, y_1), (x_2, y_2)) \in R$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$. Then the equivalence classes are $[(r, 0)] = \{(x, y) : x^2 + y^2 = r^2\}$, $r \geq 0$.


(c) Let $S = \mathbf{R}$, and $(a, b) \in R$ if $|a - b|$ is an even integer. Then the equivalence classes are $[r, r + 1)$, $\boxed{r \in [0, 2).}$

$[r] = \{2k + r : k \in \mathbb{Z}\}$

$[0] = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$ $[0.1] = \{2k + 0.1 : k \in \mathbb{Z}\}$
$[1] = \{\text{odd integers}\} = [3] = [5]$

**Proof of Theorem.** (a) Suppose $a, b \in S$.

Case (a.i) Suppose $(a, b) \in R$. We prove $[a] = [b]$. If $x \in [a]$, then $(a, x) \in R$. Because $(a, b) \in R$, we have $(b, a) \in R$ by symmetry. Hence $(b, a), (a, x) \in R$ implies that $(b, x) \in R$ by transitivity.

Similarly, we can prove $[b] \subseteq [a]$.

Case (a.ii) Suppose $(a, b) \notin R$. We prove $[a] \cap [b] = \emptyset$ by contradiction. If $x \in [a] \cap [b]$, then $(a, x) \in R$ and $(b, x) \in R$. So, $(a, x), (x, b) \in R$ implying that $(a, b) \in R$, which is a contradiction.

(b) We prove that $S$ is a disjoint union of $[a]$ with $a \in S$.

First, $[a]$ is non-empty because $a \in [a]$ by reflexive property.

Second, $[a] \cap [b] = \emptyset$ if $[a] \neq [b]$.

Third, $[a] \subseteq S$ implies $\cup_{a \in S}[a] \subseteq S$; $x \in S$ implies $x \in [x]$ so that $x \in \cup_{a \in S}[a]$. □

**Theorem** Let $P = \{A_j : j \in J\}$ be a partition of a non-empty set $A$. Define $R$ on $A$ by $xRy$ if $x, y \in A_j$ for some $j \in J$. Then $P$ is the set of equivalence classes of $A$ under $R$.

**Example** (a) The remainder classes $[0], \ldots, [n - 1]$ forms a partition of $\mathbf{Z}$.

(b) The straight lines $L_r = \{(x, y) : x + y = r\}$, $r \in \mathbf{R}$, forms partition of $\mathbf{R} \times \mathbf{R}$.

**Proof of Theorem** Suppose $P$ is a partition.

Define a relation on $R$ by $(a, b) \in R$ if $a, b \in A_j$ for some $j \in J$. such an $A_j$ exists because $P$ is a partition.

Reflexive. If $a \in A$, then $a, a \in A_j$ for some $j \in J$. So, $(a, a) \in R$.

Symmetric. If $a, b \in A$ such that $a, b \in A_j$ for some $j \in J$, then $b, a \in A_j$ and thus $(b, a) \in R$.

Transitive. If $a, b, c \in A$ such that $(a, b) \in R, (b, c) \in R$, then $a, b \in A_j, b, c \in A_k$ for some $j, k \in J$. Since $b \in A_j \cap A_k$, we see that $A_j = A_k$ so that $a, c \in A_j$. Hence, $(a, c) \in R$. □

$f \subseteq A \times B$

**Math 214 Foundations of Higher Mathematics** C.K. Li

**Chapter 9 Functions**

**Definition** let $A, B$ be non-empty sets. A function (map, mapping) $f$ from $A$ to $B$, written as $f : A \to B$, is a relation from $A$ to $B$ such that every element in $A$ is related to a **unique element** in $B$.

**Terminology** The set $A$ is the domain of $f$; $B$ is the co-domain of $f$. Range$(f) = \{b \in B : f(x) = b$ for some $x \in A\}$

**Notation** We write $f(a) = b$ if $(a, b) \in f$, and we say that $b$ is the image of $a$ under $f$, also, $f$ maps $a$ to $b$.

**Terminology** Two maps $f : A \to B$ and $g : A \to B$ are equal if $f(a) = g(a)$ for all $a \in A$.

**Notation** The set of all functions from $A$ to $B$ is the set $B^A = \{f : f$ is a function from $A$ to $B\}$.

**Examples** $\{a, b, c\}^{\{1,2\}}$, $\mathbf{R}^{\mathbf{N}}$, etc.

5 ~~fr~~

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = \frac{1}{x}$ is not a function

because $f(0)$ has no meaning,

i.e., $0 \in A = \mathbb{R}$ has no relative in $B = \mathbb{R}$.

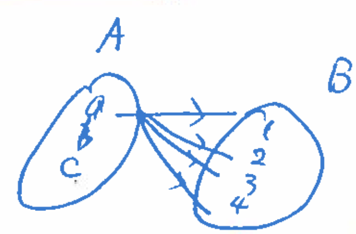
Fix the problem

Change the domain

6 $f : (\mathbb{R} - \{0\}) \to \mathbb{R}$.

$f(x) = \frac{1}{x}$ is a function

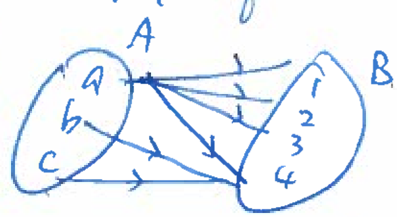
1




NOT a function

$\because$ b, c has no "relatives" in B

2




NOT a function

$a \in A$ has too many "relatives" in B

3 $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$ ✓

4 $f : \mathbb{R} \to \mathbb{R}$

$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$ ✓

§8.1 - 8.4

More generally, one can define a relation between two sets.

**Definition, notation, and terminology** Let $A$ and $B$ be sets.

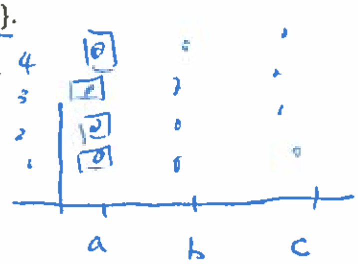
A relation $R$ from $A$ to $B$ is a subset of $A \times B$. We write $xRy$ if $(x, y) \in R$.

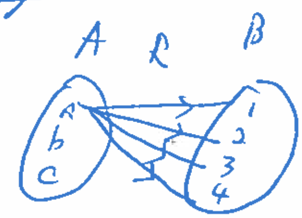
The **domain** of $R$ is $\text{dom}R = \{x \in A : (x, y) \in R \text{ for some } y \in B\}$.

The **range** of $R$ is $\text{ran}R = \{y \in B : (x, y) \in R \text{ for some } x \in A\}$.

If $|A| = m$, $|B| = n$, $|A \times B| = m \cdot n$, then there are $2^{mn}$ possible relations

**Examples** (a) Relations from $A = \{a, b, c\}$ to $B = \{1, 2, 3, 4\}$.





Domain$(R) = \{ a \}$

Range$(R) = \{1, 2, 3, 4\}$

(b) Relations from $\mathbf{Z}$ to $\{0, 1\}$.

Example: $R = \{(a,b) : a \in \mathbb{Z},\ b = \{0,1\}$
$(a, 0)$ for even $a$
$(b, 1)$ for odd $b$ $\}$

$\phi$

Dom$(R) = \mathbb{Z}$

Rang$(R) = \{0, 1\}$.

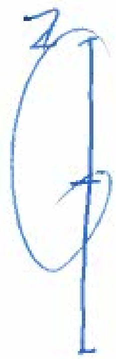
(c) Relations from $\mathbf{R}$ to $\mathbf{Z}$.

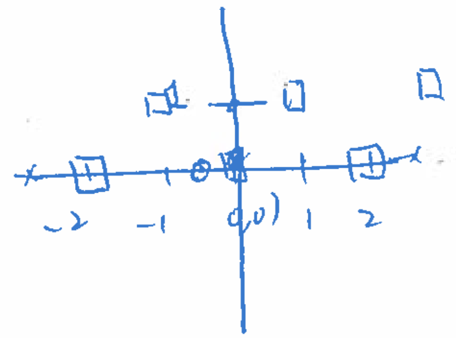
Example:

$R = \{(a, b) : a \in \mathbb{R},\ b = [a] \}$

$\phi$

The largest integer smaller than or equal to $a$

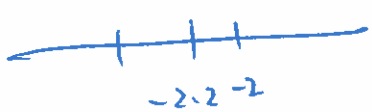




$[1.1] = [1]$  $[-2.2] = [-3]$

$[1.5] = [1]$

$[1] = [1]$



Dom$(R) = \mathbb{R}$

Range$(R) = \mathbb{Z}$

Example.

$R = \{(a,b) : a \in \mathbb{R},\ b = 1\}$

Dom$(R) = \mathbb{R}$

Range$(R) = \{1\}$.

**Definition** Let $f : A \to B$ be a function.

It is **one-to-one (injective)** provided $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$ in $A$,
i.e., if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

It is **onto (surjective)** provided the range of $f$ is $B$,
i.e., for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.

It is **bijective (one-one and onto)** if it is both injective and surjective.

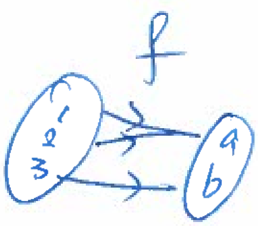
**Examples** (a) $f : \mathbf{R} \to \mathbf{R}$, ....

(b) $f : \mathbf{R} - \{2\} \to \mathbf{R}$ such that $f(x) = 3x/(x-2)$.

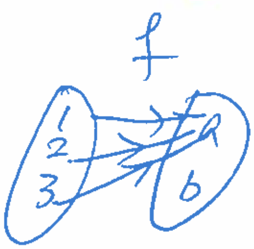
(c) $f : \mathbf{Z}_4 \to \mathbf{Z}_4$ such that $f([x]) = [3x+1]$.

(d) $f : \mathbf{Z} \to 2\mathbf{Z}$ such that $f(x) = 2x$.

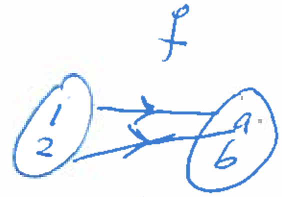
is onto



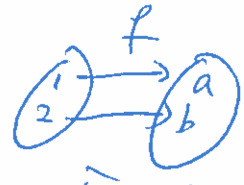
is onto



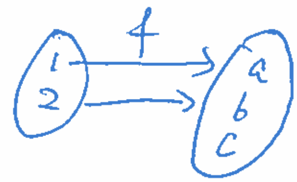
is not onto



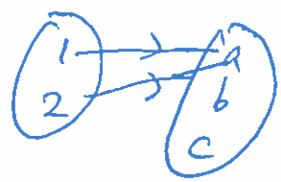
is not 1-1



is 1-1



is 1-1



is not 1-1