with the same number of elements ✓ ✓ ✓ ✓ ✓ set is a function

Theorem Suppose A and B are finite non-empty sets with same number of elements and $f : A \to B$. Then f is one-one if and only if f is onto.

~~**Remark** For $f, g : \mathbf{R} \to \mathbf{R}$, we can define $f \pm g$, fg, and f/g if $g(x)$ is never 0.~~

Remark: If X is a finite set, we can always let $X = \{x_1, \dots, x_m\}$ for some $m \in \mathbb{N}$.

Proof: Let $A = \{a_1, \dots\dots, a_n\}$.

$B = \{b_1, \dots, b_n\}$

Suppose $f : A \to B$ is a 1-1 function

Then $a_i \neq a_j \Rightarrow f(a_i) \neq f(a_j)$ in B.

$\therefore$ $\{f(a_1), \dots, f(a_n)\}$ has n distinct elements. So $f(A) = B$.

$\therefore$ f is surjective.

Suppose f is a surjection, then f is 1-1.

We prove the contrapositive. Suppose f is not 1-1.

Then there are $a_i \neq a_j$ such that $f(a_i) = f(a_j) \in B$.

So $f(A) = \{f(a_1), \dots, f(a_n)\}$ has at most $n-1$ elements. Thus $f(A) \neq B$.

$\therefore$ i.e., f is not surjective.

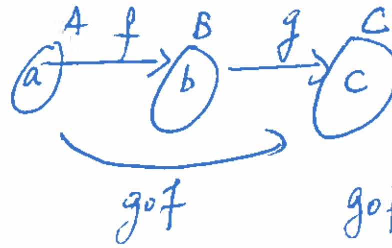
Definition If $f : A \to B$ and $g : B \to C$, then the **composite** function $h = g \circ f : A \to C$ is defined by $h(a) = g(f(a))$ for every $a \in A$,

Theorem Let $f : A \to B$ and $g : B \to C$ be functions.

(a) If f and g are one-one, then so is $g \circ f$.

(b) If f and g are onto, then so is $g \circ f$.

(c) If f and g are bijective, then so is $g \circ f$.



$g\circ f(a) = g(f(a))$
$= g(b)$
$= c.$

$h = g\circ f : A \to C$

Proof:

(a)

Assume f, g are 1-1.

If $a_1 \neq a_2$ in A then $f(a_1) \neq f(a_2)$ in B

if $b_1 \neq b_2$ in B then $g(b_1) \neq g(b_2)$ in C

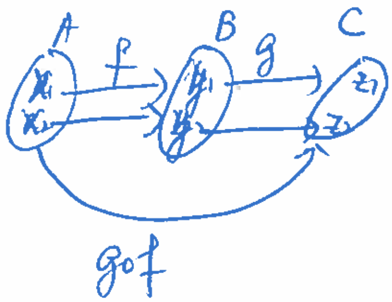
Consider $g\circ f : A \to C$

If $x_1 \neq x_2$ in A, then $g(f(x_1)) = g(y_1)$, $g(f(x_2)) = g(y_2)$ with $y_1 = f(x_1)$, $y_2 = f(x_2)$ & $y_1 \neq y_2$ as f is 1-1. $\therefore$ $g(y_1) \neq g(y_2)$ in C

~~$g\circ f(x_1) \neq g\circ f(x_2)$~~ in C

$\therefore$ $g\circ f(x_1) \neq g\circ f(x_2)$ in C

$\therefore$ $g\circ f$ is 1-1



(b) Assume f, g are onto. Then $f(A) = B$ and $g(B) = C$.

Then $g\circ f(A) = g(B) = C$. ✓.

Proof 2. For any $b \in B$, there is $a \in A$ s.t. $f(a) = b$;

for any $c \in C$ there is $b \in B$ s.t. $g(b) = c$.

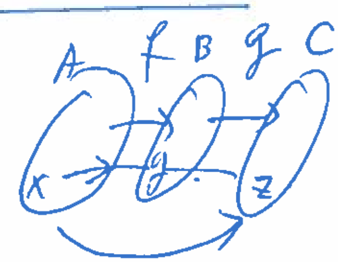
$\therefore$ For any $z \in C$.

Then there is $y \in B$ s.t. $g(y) = z$.

Also ~~Then~~ there is $x \in A$ s.t. $f(x) = y$.

$\therefore$ $g\circ f(x) = g(y) = z$.

$\therefore$ $x \in A$ such that $g\circ f(x) = z$

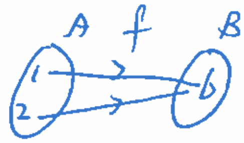


(c) By (a) & (b), the result follows.

Definition Given a relation R from A to B, we can define the **inverse relation** R^{-1} from B to A.

Theorem Let $f : A \to B$ be a function. then the inverse relation f^{-1} from B to A is a function if and only if f is bijective. In such a case, f^{-1} is also bijective.

Example



$f: \{1,2\} \to \{b\}$
s.t. $f(1)=f(2)=b$

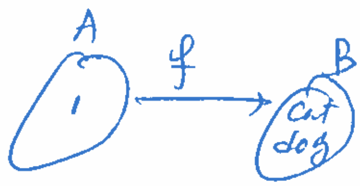
is a function, and can be written as

Then $f = \{(1,b), (2,b)\} \subseteq A \times B$

Then the inverse relation $f^{-1} = \{(b,1), (b,2)\}$
$\subseteq B \times A$

f^{-1} is NOT a function. because $b \in B$ is related to both 1 & 2 in A

Example



$f: \{1\} \to \{cat, dog\}$
$f(1) = cat$.

$f = \{(1, cat)\} \subseteq A \times B$

Then the inverse relation $f^{-1} = \{(cat, 1)\} \subseteq B \times A$.

f^{-1} is NOT a function because $dog \in B$ is not related to anything in A

Proof Assume f is 1-1 & onto. To prove the inverse relation f^{-1} from B to A is a bijection.

① To prove $f^{-1}: B \to A$ is well-defined.

Let $b \in B$. There is $a \in A$.
i.e. $(a,b) \in f$ because f is surjective
i.e. $(b,a) \in f^{-1}$.

Next. Suppose $(b,a_1) \in f^{-1}$ & $(b,a_2) \in f^{-1}$.
Then (a_1,b) $(a_2,b) \in f$
i.e. $a_1 = a_2$ because f is 1-1.

So f^{-1} is well-defined

1-1: Suppose $b_1 \neq b_2$ in B.
Consider $f^{-1}(b_1) = a_1$, $f^{-1}(b_2) = a_2$ in A

i.e., $(a_1, b_1) \in f$
$(a_2, b_2) \in f$

So $a_1 \neq a_2$ Else if $a_1 = a_2 = a$ &
$f(a) = b_1$
$f(a) = b_2$
with $b_1 \neq b_2$
!!!

$f^{-1}: B \to A$
ϕ

Onto For any $a \in A$
there is $b \in B$ s.t.
$f^{-1}(b) = a$, i.e., $(b,a) \in f^{-1}$.
Because for every $a \in A$, there is $b \in B$ s.t. $(a,b) \in f$

Now suppose f^{-1} ~~: B →~~ $\subseteq B \times A$ is
a well-defined function.

Then $(f^{-1})^{-1} = \{(a,b) : (b,a) \in f^{-1}\}$
$= f$.
is a bijection by the previous part of the proof.

Permutations of a set A are bijections from A to A. If A has n elements, we assume $\boxed{A = \{1, \ldots, n\}}$ and use a special representation.

Suppose $A = \{1, \ldots, n\}$. We use the notation S_n to denote the set of bijections from A to A.

For example, S_3 has 6 elements:

$$f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},\ f_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

$f_6^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = f_6$

$f_5^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_4$

One can find inverses, and do compositions.

$f_4 \circ f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = f_6$

$f_3 \circ f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = f_2$

$f_3 \circ f_3 \circ f_3 = \ldots = f_3$

Example $A = \{1\}$. There is only one bijection $1 \xrightarrow{f} 1$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Example $A = \{1, 2\}$. There are 2 bijections

$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

In general: $A = \{1, \ldots, n\}$ f $\begin{pmatrix} 1 & \ldots & n \\ i_1 & i_2 & \end{pmatrix}$

$n\ (n-1)\ \ldots\ 1 = n!$

choices