Math 214 QUIZ 4 Your name: Sample Solution.

1. Prove or disprove that the following are functions.

(a) $f: \mathbf{Z}_2 \to \mathbf{Z}_3$ such that $f([x]_2) = [x]_3$.

Show that $[0]_2 = [2]_2$ but $f([0]_2) \neq f([2])_2$

$[0]_2 = [2]_2$ in $\mathbb{Z}_2$

But

$f([0]_2) = [0]_3$

& $f([2]_2) = [2]_3$

so that $f([0]_2) \neq f([2]_2)$ in $\mathbb{Z}_3$.

Hence f is not well-defined.

(b) $g: \mathbf{Z}_6 \to \mathbf{Z}_4$ such that $g([x]_6) = [2x]_4$.

Show that $[x]_6 = [y]_6$ implies $g([x]_6) = g([y]_6)$.

Let $[x]_6 = [y]_6$ in $\mathbb{Z}_6$

Then $x - y = 6k$ for some $k \in \mathbb{Z}$.

Hence $2x - 2y = 2(6k) = 4(3k)$.

Thus $[2x]_4 = [2y]_4$.

i.e., $g([x]_6) = g([y]_6)$.

So g is well-defined.

2. Suppose A, B are not empty sets, $f: A \to B$ and $g: B \to A$ are functions such that $g \circ f = i_A$, the identity function. Show that f is injective and g is surjective.

~~If~~ To prove f is 1-1 by contradiction.

Suppose $a_1 \neq a_2$ in A

and $f(a_1) = f(a_2) = b \in B$.

Then $a_1 = g \circ f(a_1) = g(b)$

and $a_2 = g \circ f(a_2) = g(b)$

which is impossible.

To prove g is surjective by contradiction.

Suppose $a \in A$ is such that $g(b) \neq a \ \forall \ b \in B$.

Then if $f(a) = \hat{b} \in B$

& $a = g \circ f(a) = g(\hat{b}) \neq a$.

which is impossible.

Important definitions and examples in function theory.

Let A, B be sets, and $f \subseteq A \times B$ be a relation. $(a,b) \in f$

- $f : A \to B$ is a function if every $a \in A$ is related to one and only one element $b = f(a)$ in B.

  Examples. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x - 1$ is not a function; $f : \mathbf{Z}_4 \to \mathbf{Z}_6$ defined by $f([x]_4) = [5x]_6$ is not a function.

- The inverse relation f^{-1} is a function if and only if f is bijective, and $f^{-1} : B \to A$ is the inverse function.

  Note: The notation f^{-1} can represent the inverse relation, the inverse function if f is bijective. When f is a function, f^{-1} also represents the inverse image of $B_1 \subseteq B$ so that $f^{-1}(B_1) = \{x \in A : f(a) \in B_1\}$.

- A function $f : A \to B$ is one-one if $a_1 \neq a_2$ ensures $f(a_1) \neq f(a_2)$. Equivalently, $f(a_1) = f(a_2)$ ensures $a_1 = a_2$.

- A function $f : A \to B$ is onto if $f(A) = B$. Equivalently, for every $b \in B$ we can find $a \in A$ such that $f(a) = b$.

~~Def $f : A$~~ Suppose $f \subseteq A \times B$ is a relation

Remark: If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is a bijection.

In general, if $f \subseteq A \times B$ is a relation, then $f^{-1} = \{(b,a) ; (a,b) \in f\}$ is the inverse relation.

Suppose f is a function and $Y \subseteq B$

Then ~~$f^{-1}(B) =$~~ $f^{-1}(Y) = \{x \in A : f(x) \in Y\}$

Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 10 Cardinalities of Sets

We compare the sizes of sets, especially, **infinite sets**.

Notation Given two sets A and B, we write $|A| = |B|$ if there is a bijection between them. Also, we say that the sets have the same cardinality, or they are numerically equivalent.

Theorem Let S be a collection of sets. Define a relation on S such that $(A, B) \in R$ if there is a bijection from A to B. Then R is an equivalence relation.

A set is finite if there is a bijection from the set S to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$, and we can always write

$$S = \{s_1, \dots, s_n\}. \quad \text{or} \quad S = \phi.$$

Proof: S is a non-empty collection of sets.

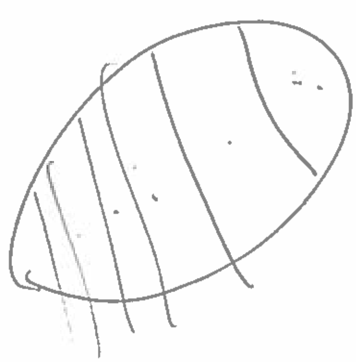
For $A, B \in S$, $(A, B) \in R$ if $|A| = |B|$ i.e., $\exists$ a bijection $f: A \to B$.

To prove R on S is an equivalence relation:

Reflexive: $\forall A \in S$ Then $i_A : A \to A$. is a bijection. $\therefore (A, A) \in R$

Symmetric: $\begin{pmatrix} \forall A, B \in S \\ (A, B) \in R \end{pmatrix}$ Then $\exists f: A \to B$, which is a bijection. Then $f^{-1}: B \to A$ is a bijection. $\therefore (B, A) \in R$

Transitive: $\begin{pmatrix} \forall A, B, C \in S \\ (A, B) \in R \\ (B, C) \in R. \end{pmatrix}$ Then $\exists$ bijections $f: A \to B$, $g: B \to C$ | Then $g \circ f : A \to C$ is bijection | $\therefore (A, C) \in R$.



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Definition A set is denumerable (or countably infinite) if $|A| = |\mathbf{N}|$. A set is countable if it is finite or it is denumerable. Otherwise, it is uncountable.

Example $\mathbf{Z}$, $2\mathbf{Z}$, etc. are denumerable.

Remark If A is finite, we may let $A = \{a_1, \ldots, a_n\}$. If A is denumerable, we may let $A = \{a_1, a_2, a_3, \ldots\}$.

Theorem If C is a subset of a denumerable set A, then one of the following holds.

(1) $C = \emptyset$. (2) $C = \{c_1, \ldots, c_n\}$ is finite. (3) C is denumerable.