Lemma: If $f_1: A_1 \to B_1$, $f_2: A_2 \to B_2$ are bijections

and $A_1 \cap A_2 = \phi$, $B_1 \cap B_2 = \phi$.

Then $f: A_1 \cup A_2 \to B_1 \cup B_2$, defined

by $f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2, \end{cases}$

is a bijection.

$f: \{0,1,2,\cdots\} \to \{1,2,\cdots\}$

Application $f: \mathbb{Q} \to \mathbb{Q} - \{0\}$.

$A_2 = \mathbb{Q} - \{0,1,2,\cdots\}$

$$\mathbb{Q} = \{0,1,2,\cdots\} \cup A_2$$

$$\downarrow f \qquad \downarrow f$$

$$\mathbb{Q} - \{0\} = \{1,2,\cdots\} \cup A_2$$

$$f(x) = \begin{cases} x+1 & \text{if } x \in \{0,1,\cdots\} \\ x & \text{if } x \in A_2 \end{cases}$$

Well-defined

$x \in \mathbb{Q}$

Case 1: $x \in \{0,1,\cdots\}$

Case 2: $x \in A_2$

$f(x) \in \mathbb{Q} - \{0\}$

Proof: Well-defined: $x \in A_1 \cup A_2$.

Case 1: $x \in A_1$, then there is a unique $b_1 \in B_1$ s.t. $f(x) = f_1(x) = b_1$.

Case 2: $x \in A_2$ then there is a unique $b_2 \in B_2$ s.t. $f(x) = f_2(x) = b_2$

$\therefore$ $b \in B_1 \cup B_2$ is a unique element s.t. $f(x) = b$

1-1. Suppose $x_1, x_2 \in A_1 \cup A_2$

$f(x_1) = f(x_2) = y \in B_1 \cup B_2$.

Case 1° $y \in B_1$. Then $x_1, x_2 \in A_1$ and $f(x_1) = f_1(x_1) = y$ & $f(x_2) = f_1(x_2) = y$. $\therefore x_1 = x_2$

Case 2° $y \in B_2$. Then $x_1, x_2 \in A_2$ and $f(x_1) = f_2(x_1) = y$, $f(x_2) = f_2(x_2) = y$ $\therefore x_1 = x_2$

In both cases. $x_1 = x_2$

Onto: Let $y \in B_1 \cup B_2$

Case 1: If $y \in B_1$ then there is $x \in A_1$ s.t. $f(x) = f_1(x) = y$

Case If $y \in B_2$ then there is $x \in A_2$ s.t. $f(x) = f_2(x) = y$.

In both cases, there is $x \in A_1 \cup A_2$ s.t. $f(x) = y$

$|\mathbb{N}| = |\mathbb{Z}| = |k\mathbb{Z}|$ $k \in \mathbb{N}$ $\quad f: \mathbb{Z} \to k\mathbb{Z}$ $f(x) = kx$ $k \in \mathbb{N}$

Definition $\boxed{\text{A set is denumerable (or countably infinite) if } |A| = |\mathbf{N}|}$ A set is countable if it is finite or it is denumerable. Otherwise, it is uncountable.

Example $\mathbf{Z}$, $2\mathbf{Z}$, etc. are $\boxed{\text{denumerable.}}$ $\quad$ ~~$\mathbb{Z} \to \mathbb{N}$~~ $\quad \mathbb{Z} \to \mathbb{N}$ $f: \mathbb{Z} \to \mathbb{N}$ $f(x) = \begin{cases} 2x & x \text{ positive}, \\ 1-2x & x \text{ negative}. \end{cases}$

Remark If A is finite, we may let $A = \{a_1, \ldots, a_n\}$. If A is denumerable, we may let $A = \{a_1, a_2, a_3, \ldots\}$.

Theorem If C is a subset of a denumerable set A, then one of the following holds.
(1) $C = \emptyset$. (2) $C = \{c_1, \ldots, c_n\}$ is finite. (3) C is denumerable.

Proof: Let $A = \{a_1, a_2, \ldots\}$. Let $C \subseteq A$

~~If $C \subseteq A$ and~~ Suppose $C \neq \emptyset$, C is not finite.

Then we have to show that

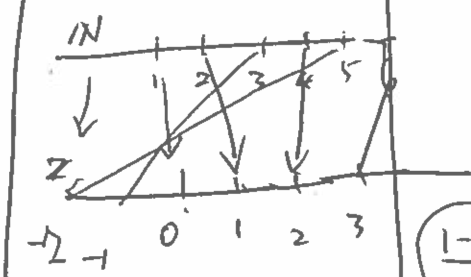
C is denumerable.

i.e., There is a bijection

$f: \mathbb{N} \to C$

$\mathbb{N} \to \mathbb{Z}$, $g: \mathbb{Z} \to \mathbb{N}$

$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{1-x}{2} & \text{if } x \text{ is odd} \end{cases}$



$\frac{1-x}{2}$

$A_j = \{a_{j1}, a_{j2}, \ldots\}$

$\boxed{f: \bigcup_{j \in \mathbb{N}} A_j \to \mathbb{N} \quad f(a_{ij}) = 2^{i-1} \cdot (2j-1). \text{ is a bijection}}$

Important consequence. In the future to prove $|A| = |\mathbb{N}|$. We need only shows there is a bijection $f: A \to S$.

set $S \subseteq \mathbb{N}$ $\quad S \neq \emptyset$, S not finite

Example: $S = \{p_j : j = 1, 2, \ldots\}$ C $\mathbb{N}$ ($p_1, p_2, \ldots p_n, \ldots$ list of primes