Let A be a set.

A is denumerable if $|A| = |\mathbb{N}|$, i.e., there is a bijection $f: A \to \mathbb{N}$ or $f: \mathbb{N} \to A$

A is countable if A is finite or A denumerable

A is uncountable if A is not finite & A is not denumerable

To prove A is denumerable, we show ...

Trick ① To show there is a bijection $f: A \to \mathbb{N}$.

Trick ② Construct $f: A \to T$ a bijection, for some $T \subseteq \mathbb{N}$, $T \neq \phi$, T is not finite.

To prove A is ~~not~~ uncountable, we show ...

Remark ① If we know $|A| = |\mathbb{N}|$ & if $|A| = |B|$

then $|B| = |\mathbb{N}|$

② ~~If~~ We know $|(0,1)| \neq |\mathbb{N}|$ is uncountable.

If $B \subseteq A$ s.t. $|B| = |(0,1)|$

then ~~|A|~~ A is uncountable.

Trick ③ Diagonal trick, By contradiction to show $\exists f: \mathbb{N} \to (0,1)$

$f(1) = 0.a_{11}\, a_{12} \cdots$

$f(2) = 0.a_{21}\, a_{22} \cdots$

$f(3) = 0.a_{31}\, a_{32}\, a_{33} \cdots$

$\vdots$

Construct $b = 0.b_1 b_2 b_3 \cdots$ with $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$, ... is not $f(m)$ $\forall m \in \mathbb{N}$

Trick ④ To show A is not countable

show that $\exists\, B \subseteq A$.

$|B| = |(0,1)|$

$|\mathbb{N}| = |\mathbb{Z}| = |k\mathbb{Z}|$ $k \in \mathbb{N}$ $f: \mathbb{Z} \to k\mathbb{Z}$ $f(x) = kx$ $k \in \mathbb{N}$

Definition A set is denumerable (or countably infinite) if $|A| = |\mathbb{N}|$. A set is countable if it is finite or it is denumerable. Otherwise, it is uncountable.

Example $\mathbf{Z}$, $2\mathbf{Z}$, etc. are denumerable.

$f: \mathbb{Z} \to \mathbb{N}$ $f(x) = \begin{cases} 2x & x \text{ positive}, \\ 1-2x & x \text{ negative} \end{cases}$

Remark If A is finite, we may let $A = \{a_1, \ldots, a_n\}$. If A is denumerable, we may let $A = \{a_1, a_2, a_3, \ldots\}$.

Theorem If C is a subset of a denumerable set A, then one of the following holds.

(1) $C = \emptyset$. (2) $C = \{c_1, \ldots, c_n\}$ is finite. (3) C is denumerable.

Proof: Let $A = \{a_1, a_2, \ldots\}$. Let $C \subseteq A$

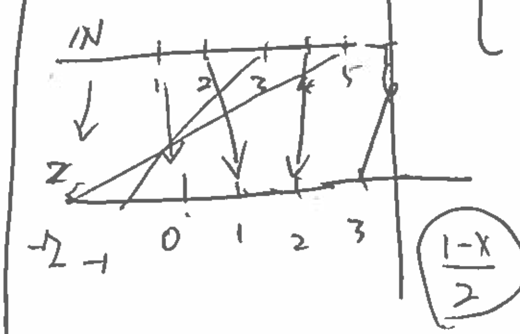
Suppose $C \neq \emptyset$, C is not finite.

Then we have to show that C is denumerable.

i.e., There is a bijection $f: \mathbb{N} \to C$

$\mathbb{N} \to \mathbb{Z}$, $g: \mathbb{Z} \to \mathbb{N}$

$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \ldots \\ \frac{1-x}{2} & \text{if } \ldots \end{cases}$



$\frac{1-x}{2}$

$A_j = \{a_{j1}, a_{j2}, \ldots\}$

$f: \bigcup_{j \in \mathbb{N}} A_j \to \mathbb{N}$ $f(a_{ij}) = 2^{i-1} \cdot (2j-1)$. is a bijection

Important consequence. In the future to prove $|A| = |\mathbb{N}|$. We need only show there is a bijection $f: A \to S$. s.t. $S \subseteq \mathbb{N}$ $S \neq \emptyset$, S not finite

Example: $S = \{p_i : i = 1, 2, \ldots\}$ ($p_1, p_2, \ldots p_n, \ldots$ list of primes

Proof: Let $A = \{a_1, a_2, \cdots \}$

Let $C \subseteq A$.

To prove one of the following holds

(1) $C = \phi$, (2) $C = \{c_1, \cdots, c_n\}$ finite or (3) ~~$|C| = \#$~~ $|C| = |\mathbb{N}|$

(1) ~~Assume~~ If $C = \phi$ or $C = \{c_1 \cdots c_n\}$ is finite nonempty

then we ~~can~~ are done

(2) Assume (1) is not true, i.e, $C \neq \phi$, C is not finite.

We have to show <u>C is denumerable</u>.

~~We~~ We will construct a bijection $f: \mathbb{N} \to C$. as follows.

$f(1) = c_{i_1} \in C$ such that $c_{i_1} = a_j$

where $j = \min\{k ; a_k \in C\}$

~~j~~ ~~a_j~~ j exists & therefore

$a_j = c_{i_1}$ exists

because of the well-ordering principle as $C \neq \phi$.

$f(2) = c_{i_2}$ such that ~~$c_{i_2} = \min$~~

$c_{i_2} = a_j$ where

$j = \min\{j : a_j \in C - \{c_{i_1}\}\}$

here j exists $\because$ C is not finite so that

$C - \{c_{i_1}\} \neq \phi$.

$\vdots$

After we define $f(k) = c_{i_k}$.

Then we set $f(k+1) = c_{i_{k+1}}$.

where $c_{i_{k+1}} = a_j$ where $j = \min\{j ; a_j \in C - \{c_{i_1}, \cdots c_{i_k}\}\}$

here j exists $\because$ $C - \{c_{i_1} \cdots c_{i_k}\} \neq \phi$.

Now $f(m)$ is defined for each m.

& only only $c_{i_m} \in C$ ~~is "related~~ equals $f(i_m)$.

One-one.

Suppose $l \neq k$ in $\mathbb{N}$

To prove $f(l) = c_{i_l}$

~~&~~ $f(k) = c_{i_k}$ are different.

Assume $l < k$.

Then $c_{i_k} = a_j$ where

$j = \min\{m : a_m \in C - \{a_{i_1}, \cdots a_{i_{k-1}}\}\}$

So $a_j \neq a_{i_l}$ . .

Onto: Let $c \in C$. Then $c = a_l$ for some $l \in \mathbb{N}$.

Consider the list ~~a_1~~ c_{i_1} c_{i_2} c_{i_3} . . c_{i_l}.

Then

$$c_{i_1} = a_{j_1}$$

$$a_l \longrightarrow \quad c_{i_2} = a_{j_2}$$

$$\vdots$$

$$c_{i_l} = a_{j_l}$$

satisfy $1 \leq j_1$, $2 \leq j_2$

$1 \leq j_1$

$2 \leq j_2$

$\vdots$

$l \leq j_l$

[$m = j_m$ happens only when $1 = j_1, 2 = j_2, \ldots, m-1 = j_{m-1}, m = j_m$]

∴ a_l will be included in this list.

∴ $f(i) = a_l = c$ for some $i \leq l$.

Theorem If A, B are denumerable sets, then so is $A \times B$.

Proof: Let $A = \{a_1, a_2, \cdots\}$, $B = \{b_1, b_2, \cdots\}$, $A \times B = \{(a_i, b_j) ;\ i, j \in \mathbb{N}\}$

Define $f : A \times B \to \mathbb{N}$

by $f(a_i, b_j) = 2^{i-1}(2j-1)$.

$f(a_1, b_1) = 2^0(2 \cdot 1 - 1) = 1$

$f(a_1, b_2) = 2^0(2 \cdot 2 - 1) = 3$

$f(a_3, b_2) = 2^{(3-1)}(2 \cdot 2 - 1)$

$= 4 \cdot 3 = 12$.

Well-defined.

If $(a_i, b_j) \in A \times B$,

then $2^{i-1}(2j-1) \in \mathbb{N}$.

1-1. Suppose $f(a_i, b_j) = f(a_r, b_s)$.

i.e., $2^{i-1}(2j-1) = 2^{r-1}(2s-1)$.

Comparing the prime factors 2 on both sides, we see that $i-1 = r-1$, i.e., $i = r$. Then $(2j-1) = (2s-1)$. $\therefore j = s$.

$\therefore (a_i, b_j) = (a_r, b_s)$

Onto. Suppose $n \in \mathbb{N}$. Then $n = 2^u(2v-1)$

Then $f(u+1, v) = n$.

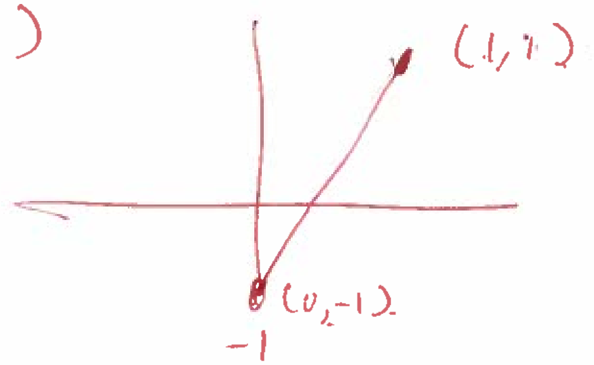
Theorem The open interval $(0, 1)$ is uncountable. ✓ Movies + blackboard discussion

Diagonal trick ! (Trick 3) .

Theorem If A is an uncountable set and $A \subseteq B$, then B is uncountable. (Trick 4)

Corollary The set of real numbers is uncountable. In fact, $|(0,1)| = |(-1,1)| = |\mathbb{R}|$.

$f: (0,1) \to (-1,1)$



$\frac{y-1}{x-1} = 2$

$y-1 = 2(x-1)$

~~$y = 2x+1$~~ $y-1 = 2x-2$

$y = 2x-1$

$f(x) = 2x-1$

Homework show $f: \mathbb{R} \to (-1,1)$

$f(x) = \frac{x}{1+|x|}$.

$|\mathbb{Q}| = |\mathbb{N}|$

$\mathbb{Q} = \{ \frac{r}{s} : r \in \mathbb{Z}, s \in \mathbb{N} \}$

$f: \mathbb{Q} \to$ [illegible] $\{ (r,s) : r \in \mathbb{Z}, s \in \mathbb{N} \} \subseteq \mathbb{Z} \times \mathbb{N}$ is denumerable