Quiz on Thursday will be from Hwk 9/10

Theorem The open interval $(0,1)$ is uncountable. ✓ Movies + blackboard discussion

Diagonal trick! (Trick 3).

$(0,1)$ is not finite nor denumerable

Clearly $(0,1)$ is not finite. e.g. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ are infinitely many elements in $(0,1)$

There is no bijection from $f: \mathbb{N} \to (0,1)$, i.e. $(0,1)$ not denumerable.

Proof by contradiction.

Suppose $\exists$ a surjection $f: \mathbb{N} \to (0,1)$.

Then we can "list" all the elements.

$$f(1) = 0.a_{11}a_{12}\cdots a_{1n}\cdots \in (0,1)$$

$$f(2) = 0.a_{21}a_{22} \quad \in$$

$$f(3) = 0.a_{31}\cdots \quad \in (0,1)$$

$$\vdots \qquad \vdots$$

$$f(k) = 0.a_{k1}a_{k2}\cdots a_{kk}\cdots$$

Consider $b = 0.b_1 b_2 b_3 \cdots \in (0,1)$

s.t. $b_1 \neq a_{11}$ $\quad b_i \neq 9$

$b_2 \neq a_{22}$

$b_3 \neq a_{33}$.

$\vdots \quad \vdots$

Then $b \neq f(k)$ for any $k \in \mathbb{N}$

because $b_k \neq a_{kk}$. $\therefore$ a contradiction.

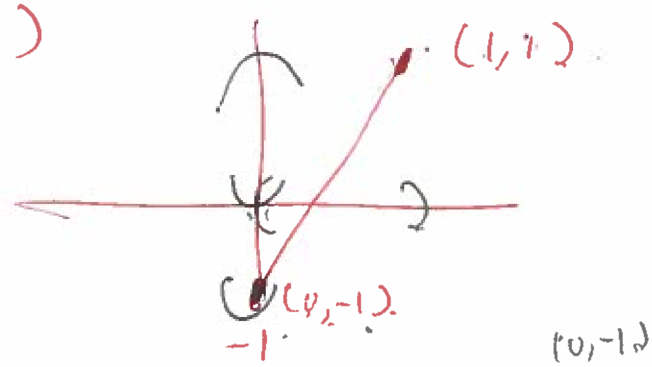
Rmk: In Homework: $S = \{(a_1, a_2, \cdots) : a_i \in \{0,1\}\}$

If B is ~~denumer~~ countable, then any subset of B is countable.

Theorem If A is an uncountable set and $A \subseteq B$, then B is uncountable. (Trick 4)

Corollary The set of real numbers is uncountable. In fact, $|(0,1)| = |(-1,1)| = |\mathbb{R}|$.

$f : (0,1) \to (-1,1)$



$\boxed{\frac{y-1}{x-1} = 2}$

$y-1 = 2(x-1)$

$\cancel{y = 2x+1}$

$y-1 = 2x-2$

$\boxed{y = 2x-1}$

$\boxed{f(x) = 2x-1}$

(0,-1)

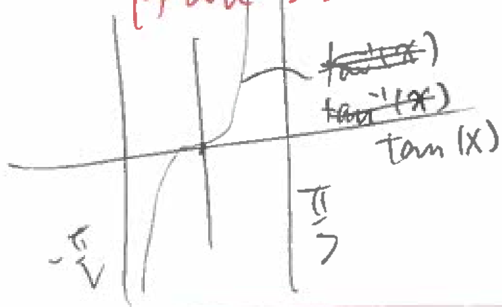
$f(0) = -1$ (1,1)

$f(1) = 1$

$f : (0,1) \to (-1,1)$ a bijection

$\therefore |(0,1)| = |(-1,1)|$

Here show



$f : \mathbb{R} \to (-1,1)$

$f(x) = \frac{x}{1+|x|}$ ✓ ✓✓

$|\mathbb{Q}| = |\mathbb{N}|$

$\mathbb{Q} = \{ \frac{r}{s} : r \in \mathbb{Z}, s \in \mathbb{N} \}$

$f : \mathbb{Q} \to \cancel{T} \cancel{S} \ \{ (r,s) : r \in \mathbb{Z}, s \in \mathbb{N} \} \subseteq \mathbb{Z} \times \mathbb{N}$ is denumerable

Proof of $|\mathbb{Q}| = |\mathbb{N}|$

Construct a bijection $f: \mathbb{Q} \to \mathbb{N}$. (See textbook).

Here we construct $f: \mathbb{Q} \to T$. $T \subseteq \mathbb{N}$ an infinite subset.

Note that $\mathbb{Q} = \left\{ \frac{n}{m} : n \in \mathbb{Z},\ m \in \mathbb{N} \right\}$

~~may~~

$$= \left\{ \frac{n}{m} : \begin{array}{l} n \in \mathbb{N},\ m \in \mathbb{N} \\ \gcd(n,m)=1 \end{array} \right\}$$

$$\cup \{0\}$$

$$\cup \left\{ \frac{-n}{m} : \begin{array}{l} n \in \mathbb{N},\ m \in \mathbb{N} \\ \gcd(n,m)=1 \end{array} \right\}$$

Define

$$\left.\begin{array}{l} f\left(\frac{n}{m}\right) = 2^n 3^m \\ f(0) = 7 \\ f\left(\frac{-n}{m}\right) = 11^n 13^m \end{array}\right\} \quad \begin{array}{l} f: \mathbb{Q} \to \mathbb{N} \\ \text{1-1 function} \end{array}$$

Let $T = f(\mathbb{Q}) = \{ f(x) : x \in \mathbb{Q} \}$

$\therefore\ |\mathbb{Q}| = |T|$ $\quad T \subseteq \mathbb{N}$ is infinite

$= |\mathbb{N}|$. $\quad$ e.g. $f\left(\frac{n}{1}\right) = 2^n \in T$ $\forall\, n = 1, 2, 3.$

e.g. $S = \{ (a,b) : \frac{a+b}{2} \in \mathbb{Q} , a < b \text{ in } \mathbb{R}. \}$

$\underset{\text{Interval}}{\varphi}$

$A = \{ (-r, r) : \cancel{r \in (0,1)} \} \subseteq S.$

$0 < r < 1$

$f: A \to (0,1) \quad f((-r,r)) = r$

$\therefore |A| = |(0,1)|$ is uncontable

Proof of Corollary

$f: [0,1] \to [0,2]$

$f(x) = 2x$

$|(0,1)| = |(-1,1)|$

$|(-1,1)| = \mathbb{R}$

Definition Let A, B be sets. If there is an injection from A to B, we write $|A| \le |B|$.

We write $|A| < |B|$ if there is an injection from A to B, but not bijection from A to B.

Continuum Hypothesis Let $|\mathbf{N}| = \aleph_0$ and $|\mathbf{R}| = c$ (the continuum).

There is no set S such that $\aleph_0 < |S| < c$. Can not determined.

Theorem Let A be a non-empty set, and 2^A be the set of functions from A to $\{0,1\}$. Then $|\mathcal{P}(A)| = |2^A|$.

$2^A = \{ f : f \text{ is a function from } A \text{ to } \{0,1\} \}$

i.e. $f(a)$ is 0 or $f(a)=1$ for every $a \in A$.

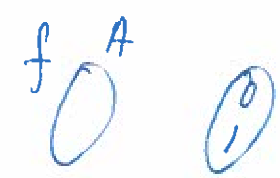
Claim: $|\mathcal{P}(A)| = |2^A|$.

$\mathcal{P}(A) = \{ S : S \text{ is a subset of } A \}$

Construct $\phi : 2^A \to \mathcal{P}(A)$

So that $\phi(f) = S$.

where $S = \{ x \in A : f(x) = 1 \}$



Well-defined: Every $f : A \to \{0,1\}$ in 2^A. yields a subset $S \in \mathcal{P}(A)$

1-1. Suppose $f_1 \ne f_2$ in 2^A.

i.e., there is $a \in A$ s.t. $f_1(a) \ne f_2(a)$,

Say $f_1(a) = 0$, $f_2(a) = 1$. $\therefore$ $a \in \phi(f_2)$, $a \notin \phi(f_1)$.

$\therefore$ $\phi(f_1) = \{ x \in A : f_1(x) = 1 \}$

$\phi(f_2) = \{ x \in A : f_2(x) = 1 \}$ are different

Onto: Let $S \subseteq A$.

Define $f : A \to \{0,1\}$

$\phi(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases}$