$$|\underline{2^A}| = |\underline{\mathcal{P}(A)}|$$

$$2^A = \left\{ f : \begin{array}{l} f \text{ is a function} \\ \text{from } A \text{ to } \{0,1\} \end{array} \right\}$$

Need to construct a bijection from 2^A to $\mathcal{P}(A)$.

Consider

$$\underline{\phi} : \underline{2^A \to \mathcal{P}(A)}$$

For $f \in 2^A$, define
$\underline{\phi(f)} = S$, where $S = \{x \in A : f(x) = 1\}$

Then every f will be 'related', 'mapped', 'sent' to a unique $S \in \mathcal{P}(A)$. So ϕ is well-defined.

1-1: If $f_1 \neq f_2$, ~~then~~ then there is $x \in A$ s.t. $\underline{f_1(x) \neq f_2(x)}$, say, $\underline{f_1(x) = 0, f_2(x) = 1}$.
(If $f_1(x) = 1$, $f_2(x) = 0$, we can interchange the roles of f_1 & f_2.)

Then $\phi(f_1) = S_1$ will not contain x
$\phi(f_2) = S_2$ contains x.

Onto: For every $S \in \mathcal{P}(A)$

Consider $f : A \to \{0,1\}$ $\quad f(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$

Then $\phi(f) = S$.

set of functions from $\mathbb{N} \to \{0,1\}$.

set of subsets of $\mathbb{N}$

Theorem We have $|2^{\mathbf{N}}| = |\mathcal{P}(\mathbf{N})| = |\mathbf{R}|$.

We will show $|2^{\mathbb{N}}| = |(0,1)|$.

First, for every $f: \mathbb{N} \to \{0,1\}$, I write f as

$(f(1), f(2), f(3), \dots)$ as a $(0,1)$ sequence.

$2^{\mathbb{N}} = \{(f(1), f(2), \dots) : f(i) \in \{0,1\}\}$

$(f(1) f(2) f(3) \dots)$

Define $\phi: 2^{\mathbb{N}} \to (0,1)$ by

$\phi((a_1 a_2 a_3 \cdots)) = (0.a_1 a_2 \cdots)_2$.

Binary numbers

$11 = (1011)_2$

$123 = 1\times 10^2 + 2\times 10^1 + 3\times 10^0$

$(a_{10} \cdots a_2 a_1 a_0)_2$

$(10101)_2 = 2^4 + 2^2 + 2^0 = 16+4+1 = 21$

$(1102)_3 = 1\times 3^3 + 1\times 3^2 + 0\times 3^1 + 2 = 27 + 9 + 2 = (38)_{10}$

$(123TE)_{12} = 1\times 12^4 + 2\times 12^3 + 3\times 12^2 + 10\times 12 + 11\times 1$

$T=10$, $E=11$

$0.123 = 1\times\frac{1}{10} + 2\times\frac{1}{100} + 3\times\frac{1}{10^3}$

$(0.01001)_2 = 2^{-2} + 2^{-5} = \frac{1}{4} + \frac{1}{32}$.

Theorem Let A be a non-empty set. Then $|A| < |\mathcal{P}(A)|$.