Math 214 QUIZ 5 Your name: Sample solution.

1. Let $\mathbf{Q}$ be the set of rational numbers. Construct with proof a bijection from $\mathbf{Q}$ to $\mathbf{Q} - \{0\}$.

Define $f_1 : \{0,1,2,\dots\} \to \{1,2,\dots\}$ by $f_1(x) = x+1$.

Then $f_1(x) \in \mathbb{N}$ for every $x \in \{0,1,\dots\}$.

$f_1(x) = f_1(y)$ means $x+1 = y+1$, $\therefore x = y$.

For every $y \in \mathbb{N}$, $x = y-1 \in \{0,1,2,\dots\}$ & $f(x) = y$

Define $f_2 : \mathbb{Q} - \{0,1,2,\dots\} \to \mathbb{Q} - \{0,1,2,\dots\}$ by $f_2(x) = x$. is a bijection

So $f(x) = \begin{cases} f_1(x) & \text{if } x \in \{0,1,\dots\} \\ f_2(x) & \text{if } x \in \mathbb{Q} - \{0,1,\dots\} \end{cases}$ from $\mathbb{Q}$ to $\mathbb{Q} - \{0\}$

is a bijection. by the Divide & Conquer Lemma

2. Let $\mathbf{N}$ be the set of natural numbers. Construct with proof a bijection from A to $\mathbf{N}$, where

$$A = \{2^i : i \in \mathbf{N}\} \cup \{3^j : j \in \mathbf{N}\}.$$

$g : A \to \mathbb{N}$

$g(x) = \begin{cases} 2i & x = 2^i \\ 2j-1 & x = 3^j \end{cases}$

Define $f_1 : \{2^i : i \in \mathbb{N}\} \to \{2i : i \in \mathbb{N}\}$

by $f(2^i) = 2i$, $i \in \mathbb{N}$

Define $f_2 : \{3^j : j \in \mathbb{N}\} \to \{2j-1 : j \in \mathbb{N}\}$

$f_2(3^j) = 2j-1$, $j \in \mathbb{N}$.

For f_1 ~~$f_1(x) \in 2^i$~~ well-defin: for every $2^i \in \{2^i : i \in \mathbb{N}\}$ $f(2^i) = 2i \in \{2i : i \in \mathbb{N}\}$

1-1: If $2^i \neq 2^j$ then $f(2^i) = 2i$ $f(2^j) = 2j$ are different

onto: For any $2i \in \{2i : i \in \mathbb{N}\}$, $f(2^i) = 2i$

For f_2: For every $3^j \in \{3^j : j \in \mathbb{N}\}$, $f(3^j) = 2j-1 \in \{2j-1 : j \in \mathbb{N}\}$

If $3^i \neq 3^j$, then $f(3^i) = 2i-1$, $f(3^j) = 3j-1$ are different.

If $2j-1 \in \{2j-1 : j \in \mathbb{N}\}$, then $f(3^j) = 2j-1$.

By Divide & Conquer Lemma, $f : A \to \mathbb{N}$ defined by

$f(x) = \{ f_1(x)$ if $x = 2^i$, $i \in \mathbb{N}$ is a bijection

Theorem $\quad |2^{\mathbb{N}}| = |\mathbb{R}|$

Binary arithmetic.

Every number in $(0,1)$ can be written as.

$$(0.a_1a_2a_3\cdots)_2 = a_1\times\frac{1}{2} + a_2\frac{1}{4} + a_3\cdot\frac{1}{8} + \cdots + a_k\cdot\bot$$

$$\boxed{\begin{aligned}\frac{3}{4} &= \frac{1}{2}+\frac{1}{4}\\ &= (0.110\cdots0)_2\end{aligned}}$$

The set $(0,1)$ has the same cardinality as

$$S_1 = \{(0.a_1\cdots a_n,\cdots)_2 : a_i \in \{0,1\}\}.$$

which has the same cardinality

$$S_2 = \{(a_1, a_2, \cdots, a_n, \ldots) : a_i \in \{0,1\}\}$$

which has cardinality ~~as~~ uncountable.

and

$$S_3 = \{(f(1), f(2), \cdots, f(n), \ldots) : f : \mathbb{N} \to \{0,1\}\}$$

set of functions from $\mathbb{N}$ to $\{0,1\}$

$$\therefore \quad |S_3| = |S_2| = |S_1| = |(0,1)| = |\mathbb{R}|$$

Theorem. Let A be a set. Then $|A| < |\mathcal{P}(A)|$.

i.e. There is an injection $f: A \to \mathcal{P}(A)$.

There is no ~~sur~~ bijection / surjection from A to $\mathcal{P}(A)$.

Proof: If $A = \phi$, $\mathcal{P}(A) = \{\phi\}$, there is no ~~injection~~ surjection from A to $\mathcal{P}(A)$

The empty relation / function can be viewed as an injection.

Assume $A \neq \phi$.

Consider $f: A \to \mathcal{P}(A)$, $f(x) = \{x\} \in \mathcal{P}(A)$.

It is a well-defined because $x \in A$ ensures $f(x) = \{x\} \in \mathcal{P}(A)$

1-1: $x \neq y$ in A, $f(x) = \{x\} \neq \{y\} = f(y)$ in $\mathcal{P}(A)$

$\therefore |A| \leq |\mathcal{P}(A)|$

To finish the proof, we show that there is no surjection $g: A \to \mathcal{P}(A)$

It follows that there is no bijection $f: A \to \mathcal{P}(A)$.

By contradiction, assume that there is a surjection $g: A \to \mathcal{P}(A)$.

Then for every $a \in A$, there is a subset $g(a) \in \mathcal{P}(A)$.

Consider $B = \{x \in A : x \notin g(x)\}$.

Claim: There is no $b \in A$ s.t. $g(b) = B$.

By contradiction. Assume $b \in A$ satisfies $g(b) = B$.

Consider 2 cases.

Case 1° $b \in B$. Then $b \in g(b) = B$, $\therefore b \notin B$ by definition. $\cancel{\phi}$

Case 2° $b \notin B$. Then $b \notin B = g(b)$. $\therefore x \in B$ by definition of B $\cancel{\phi}$

Remark Let A, B be sets. Exactly one of the following holds. $|A| = |B|$, $|A| < |B|$, $|A| > |B|$.

The Schröder-Bernstein Theorem Let A, B be sets. If $|A| \le |B|$ and $|B| \le |A|$, then $|A| = |B|$.

We use the following the lemma proved in your homework.

Lemma (Divide and Conquer) If $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are bijective (injective, surjective) and $A_1 \cap A_2 = \emptyset = B_1 \cap B_2$, then $f : A_1 \cup A_2 \to B_1 \cup B_2$ defined by $f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2, \end{cases}$ is bijective (injective, surjective).

Next, we prove the following.

Theorem 10.19 Suppose A and B are non-empty sets such that $B \subseteq A$. If there is an injection from $f : A \to B$, then there is a bijection from A to B.

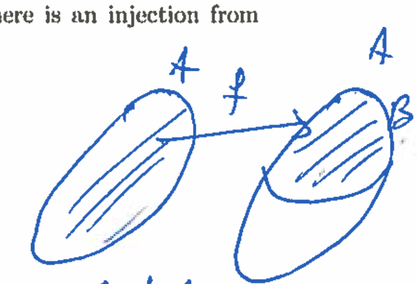
and $f(A) = B$: then we get a bijection

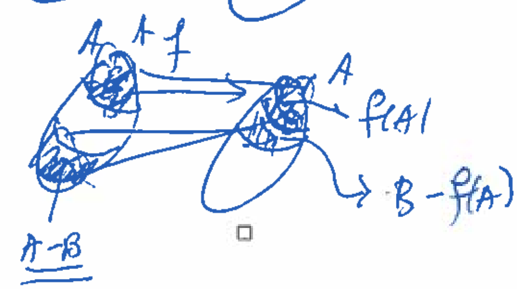
Proof. If $A = B$, then we are done. So, we assume that $A - B \neq \emptyset$.
If f is bijective, then we are done. So, we assume that $B - f(A) \neq \emptyset$.
Consider $\boxed{B' = \{f^n(x) : x \in A - B, n \in \mathbf{N}\}}$
Then $B' \subseteq f(A)$.
Moreover, for any $x \in A - B$, $f(x) \in B'$, $f^2(x) \in B'$, $f^3(x) \in B'$, etc.
Then $h_1 : (A - B) \cup B' \to B'$ defined by $h(x) = f(x)$ is bijective,
and the identity map on $D = B - B'$ is bijective.
So, $h : A \to B$ defined by

$$h(x) = f(x) \text{ if } x \in (A - B) \cup B',$$

$$h(x) = x \text{ otherwise}$$

is a bijection. $\square$





Proof of Schröder-Bernstein Theorem

Assume that $f : A \to B$ and $g : B \to A$ are injections.

Then $g(B) \subseteq A$ and $g \circ f(A) = C \subseteq A$, where $g \circ f$ is injective. So, there is a bijection from A to C. Now, $g^{-1} : g(B) \to B$ is a bijection. So, $g^{-1} \circ h : A \to B$ is a bijection. $\square$