Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 1 Sets

Definition A set is a collection of objects, its elements (or members).

Notation

We use curly bracket { } to embrace the elements in the set,

elements are separated by comma ',' in the brackets,

$x \in S$: x is an element (member) of S, or x belongs to the set S;

$x \notin S$: x is not an element (member) of S, x does not belong to the set S.

$|S|$: Carlinatily (size, or cardinal number) of a set S, which may be finite or infinite.

A set is finite if $|S| = n$ for some nonnegative integer n.

1.1 Describing a set by (a) listing all elements; (b) describing the property.

$A = \{a, e, i, o, u\}$; A is the set of vowels; $a \in A; a, e \in A, m \notin A$; $|A| = 5$.

$B = \{\text{cat, dog, pig}\}$ is the set consisting of the elements: cat, dog, pig; $|B| = 3$.

$C = \{1, 2, \{1, 3\}, \text{cat}\}$; $\{1, 3\} \in C; \text{cat} \in C; 1 \in C; 3 \notin C$.

$D = \{-2, 2\} = \{x : x \text{ is a real number such that } x^2 = 4\}$
$= \{x : x \text{ is a real number such that } |x| = 2\}$.

$E = \{1, 3, 5, 7\} = \{x : (x-1)(x-3)(x-5)(x-7) = 0\}$;
E is the set of odd integers between 0 and 8.

Remark Two sets A and B are equal, denoted by $A = B$, if they contain the same elements.
That is, every element in A is an element in B, and vice versa.

Example $\{1, 2, 3\} = \{1, 3, 2\} = \{1, 2, 2, 3\}$.

Special sets

Empty set $\emptyset$. It is also called the null set or the void set. Note that $|\emptyset| = 0$.

Example Suppose $F = \{\emptyset, \{\emptyset\}\}$.

(a) $\emptyset \in F$? (b) $\{\emptyset\} \in F$? (c) $|F| =$???

We always assume that there is a **universal** set U containing all the objects under consideration.

It will lead to a (Russell) paradox if we assume that there is a set containing EVERYTHING.

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$ is the set of natural numbers.

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

$\mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\} = \{a/b : a, b \in \mathbb{Z}, b \neq 0\}$ is the set of rational numbers.

$\mathbb{R}$ is the set of real numbers.

$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$ is the set of complex numbers.

Remark All of the above sets are infinite, say, $|\mathbb{N}|$ is infinite, or we write $|\mathbb{N}| = \infty$.

1.2 Subsets

Definition A set X is a subset of Y if every element in X is an element in Y, denoted by $X \subseteq Y$.

If in addition that $X \neq Y$, then X is a proper subset of Y.

Example $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4, 5\}, Z = \{4\}$.

Example $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Remark Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$.

Definition The set of all subsets of X is the power set of X, denoted by $\mathcal{P}(X)$.

Example (a) $X = \emptyset$. (b) $Y = \{1\}$. (c) $Z = \{0, \emptyset, \{\emptyset\}\}$.

Distinction between subsets and memberships

Example Suppose $F = \{\emptyset, \{\emptyset\}\}$.

1.3 Set operations and Venn diagrams

Venn diagrams can help depict the relationships and operations on sets.

Definition Let X and Y be sets.

- Their union, denoted by $X \cup Y$, is the set $\{x \in U : x \in X$ **or** $x \in Y\}$.
- Their intersection, denoted by $X \cap Y$, is the set $\{x \in U : x \in X$ **and** $x \in Y\}$.
- The complement of X in the universal set U, denoted by $\overline{X}$, is the set $\{x \in U : x \notin X\}$.
- The relative complement of X in Y, denoted by $Y - X$, is the set $\{x \in U : x \in Y, x \notin X\}$.

Example $X = \{1, 2, 3\}, Y = \{1, 2, 3, 4, 5\}, Z = \{4\}$.

Recall the notation of intervals of real numbers.

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

$[a, b) =$

$(a, b] =$

$(a, b) =$

Example $A = [1, 5], B = [2, 6), C = \{1, 2, 3, 4, 5, 6\}$.

1.4 Indexed collections of sets

We may consider a family of sets A_j with j lying in an index set J.

Then we can consider their union, intersections, etc.

Example Let $A_r = [0, r]$ with $r > 0$; $B_r = \{0, r\}$.

One may determine $\cap_{r \in R} A_r, \cup_{r \in R} A_r$, say, with $R = \{1, 2, 3\}$.

1.5 Partition of sets

A partition of a set X is a collection of pairwise disjoint **nonempty** subsets whose union is X.

Examples $X = \{1, 2\}$ and $Y = \{\phi, \{\phi\}\}$.

1.6 Cartesian products of sets

Definition Let A, B be sets. Their Cartesian product is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

Examples $\mathbb{N} \times \mathbb{R}$ and $[1, 2] \times [3, 4]$.