Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 6 Mathematical Induction

We develop a machinery to show that the open statement $P(n)$ is true for all natural numbers (or all natural numbers starting from n_0).

6.1 The principle of mathematical induction

A set of real number may or may not have a **least** element.

Theorem If the least number exits in a set of real number, then it is unique.

The well ordering principle of natural numbers Every non-empty subset of $\mathbf{N}$ has a least element.

Remarks The well ordering principle fails for $\mathbf{Q}$ or $\mathbf{R}$.

One can prove the well ordering principle for other subsets of $\mathbf{Z}$. For example, the set of integers larger than -1000, the set of nonnegative even integers.

The principle of mathematical induction Let $P(n)$ be an open statement with $n \in \mathbf{N}$. Suppose we can established the following two statements.

(a) $P(1)$ is true. (b) If $P(k)$ is true then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{N}$.

Remark This is the domino effect!

Proof. Show that $S = \{n \in \mathbf{N} : P(n) \text{ is false}\}$ no smallest element. So, it is empty.

Examples (a) $P(n)$: $1 + \cdots + n = n(n+1)/2$.

(b) $P(n)$: $1^2 + \cdots + n^2 = n(n+1)(2n+1)/6$.

Examples (c) $P(n) : \frac{1}{2\cdot 3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$.
(d) $P(n) : 4|(5^n - 1)$.
(e) $P(n) : 6|(n^3 - n)$.
[We can prove (e) by the method of minimum counter example. See Section 6.3.]

6.2/6.4 More general principles

Principle of MI - 2 Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$. (b) For $k \geq m$, if $P(k)$ is true then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

Remark This follows from the fact that $S = \{n \in \mathbf{Z} : n \geq m\}$ is well ordered.

Examples (a) For every $n \geq 5$, $2^n > n^2$.
(b) Let $A_1, \ldots, A_n$ be sets with $n \geq 2$. Then $\overline{\cup_{i=1}^n A_i} = \cap_{i=1}^n \overline{A_i}$.
(c) Suppose $n \geq 0$. If a set with n elements then its power set has 2^n elements.

Principle of MI - 3 Suppose we can establish the statements.

(a) $P(m)$ is true for a certain $m \in \mathbf{Z}$.

(b) For $k \geq m$ if $P(j)$ is true for all $j = m, \ldots, k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbf{Z}$ with $n \geq m$.

Remark We use all the previous fallen dominoes to ensure the next domino would fall also.

Example Every integer $n \geq 2$ is a prime or a product of primes.

Recursive sequences Induction is useful in proving formulas and properties of recursive sequences, i.e., sequences $\{a_1, a_2, \ldots\}$ defined by specification of $a_1, \ldots, a_k$ and a relation/formula expressing a_m in terms of the previous $a_1, \ldots, a_k$, for $m > k$.

Examples (a) Prove that $a_n = n^2$ for all $n \in \mathbf{N}$ if the sequence $\{a_1, a_2, \ldots\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 4,\ \text{and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for } n \geq 3.$$

(b) Find a formula for a_n with a proof if the sequence $\{a_1, a_2, \ldots\}$ is defined recursively by

$$a_1 = 1,\ a_2 = 2,\ \text{and } a_n = a_{n-1} + 2a_{n-2} \text{ for } n \geq 3.$$