Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 9 Functions

Definition let A, B be non-empty sets. A function (map, mapping) f from A to B, written as $f : A \to B$, is a relation from A to B such that every element in A is related to **a unique element** in B.

Terminology The set A is the domain of f, B is the co-domain of f.

Notation We write $f(a) = b$ if $(a, b) \in f$, and we say that b is the image of a under f, also, f maps a to b.

Terminology Two maps $f : A \to B$ and $g : A \to B$ are equal if $f(a) = g(a)$ for all $a \in A$.

Notation The set of all functions from A to B is the set $B^A = \{f : f \text{ is a function from } A \text{ to } B\}$.

Examples $\{a, b, c\}^{\{1,2\}}$, $\mathbf{R}^{\mathbf{N}}$, etc.

Definition Let $f : A \to B$ be a function.

It is **one-to-one (injective)** provided $f(a_1) \neq f(a_2)$ whenever $a_1 \neq a_2$ in A,
i.e., if $f(a_1) = f(a_2)$ then $a_1 = a_2$.

It is **onto (surjective)** provided the range of f is B,
i.e., for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.

It is **bijective (one-one and onto)** if it is both injective and surjective.

Examples (a) $f : \mathbf{R} \to \mathbf{R}$, ...

(b) $f : \mathbf{R} - \{2\} \to \mathbf{R}$ such that $f(x) = 3x/(x-2)$.

(c) $f : \mathbf{Z}_4 \to \mathbf{Z}_4$ such that $f([x]) = [3x+1]$.

(d) $f : \mathbf{Z} \to 2\mathbf{Z}$ such that $f(x) = 2x$.

Theorem Suppose A and B are finite non-empty sets with same number of elements, and $f : A \to B$. Then f is one-one if and only if f is onto.

Remark For $f, g : \mathbf{R} \to \mathbf{R}$, we can define $f \pm g$, fg, and f/g if $g(x)$ is never 0.

Definition If $f : A \to B$ and $g : B \to C$, then the **composite** function $h = g \circ f : A \to C$ is defined by $h(a) = g(f(a))$ for every $a \in A$,

Theorem Let $f : A \to B$ and $g : B \to C$.

(a) If f and g are one-one, then so is $g \circ f$.

(b) If f and g are onto, then so is $g \circ f$.

(c) If f and g are bijective, then so is $g \circ f$.

Definition Given a relation R from A to B, we can define the **inverse relation** R^{-1} from B to A.

Theorem Let $f : A \to B$ be a function. then the inverse relation f^{-1} from B to A is a function if and only if f is bijective. In such a case, f^{-1} is also bijective.

Permutations of a set A are bijections from A to A. If A has n elements, we assume $A = \{1, \ldots, n\}$, and use a special representation.

Suppose $A = \{1, \ldots, n\}$. We use the notation S_n to denote the set of bijections from A to A.

For example, S_3 has 6 elements:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

One can find inverses, and do compositions.

Important definitions and examples in function theory.

Let A, B be sets, and $f \subseteq A \times B$ be a relation.

- $f : A \to B$ is a function if every $a \in A$ is related to one and only one element $b = f(a)$ in B.

  Examples. $f : \mathbf{N} \to \mathbf{N}$ defined by $f(x) = x-1$ is not a function; $f : \mathbf{Z}_4 \to \mathbf{Z}_6$ defined by $f([x]_4) = [5x]_6$ is not a function.

- The inverse relation f^{-1} is a function if and only if f is bijective, and $f^{-1} : B \to A$ is the inverse function.

  Note: The notation f^{-1} can represent the inverse relation, the inverse function if f is bijective. When f is a function, f^{-1} also represents the inverse image of $B_1 \subseteq B$ so that $f^{-1}(B_1) = \{x \in A : f(a) \in B_1\}$.

- A function $f : A \to B$ is one-one if $a_1 \neq a_2$ ensures $f(a_1) \neq f(a_2)$. Equivalently, $f(a_1) = f(a_2)$ ensures $a_1 = a_2$.

- A function $f : A \to B$ is onto if $f(A) = B$. Equivalently, for every $b \in B$ we can find $a \in A$ such that $f(a) = b$.