Math 214 Foundations of Higher Mathematics C.K. Li

Chapter 10 Cardinalities of Sets

We compare the sizes of sets, especially, **infinite sets**.

Notation Given two sets A and B, we write $|A| = |B|$ if there is a bijection between them. Also, we say that the sets have the same cardinality, or they are numerically equivalent.

Theorem Let S be a collection of sets. Define a relation on S such that $(A, B) \in R$ if there is a bijection from A to B. Then R is an equivalence relation.

Definition A set is denumerable (or countably infinite) if $|A| = |\mathbf{N}|$. A set is countable if it is finite or it is denumerable. Otherwise, it is uncountable.

Example $\mathbf{Z}$, $2\mathbf{Z}$, etc. are denumerable.

Remark If A is finite, we may let $A = \{a_1, \ldots, a_n\}$. If A is denumerable, we may let $A = \{a_1, a_2, a_3, \ldots\}$.

Theorem If C is a subset of a denumerable set A, then one of the following holds.
(1) $C = \emptyset$. (2) $C = \{c_1, \ldots, c_n\}$ is finite. (3) C is denumerable.

Theorem If A, B are denumerable sets, then so is $A \times B$.

Theorem The open interval $(0, 1)$ is uncountable.

Theorem If A is an uncountable set and $A \subseteq B$, then B is uncountable.

Corollary The set of real numbers is uncountable. In fact, $|(0,1)| = |(-1,1)| = |\mathbf{R}|$.

Definition Let A, B be sets. If there is an injection from A to B, we write $|A| \leq |B|$.

We write $|A| < |B|$ if there is an injection from A to B, but not bijection from A to B.

Continuum Hypothesis Let $|\mathbf{N}| = \aleph_0$ and $|\mathbf{R}| = c$ (the continuum).

There is no set S such that $\aleph_0 < |S| < c$.

Theorem Let A be a non-empty set, and 2^A be the set of functions from A to $\{0, 1\}$. Then $|\mathcal{P}(A)| = |2^A|$.

Theorem We have $|2^{\mathbf{N}}| = |\mathcal{P}(\mathbf{N})| = |\mathbf{R}|$.

Theorem Let A be a non-empty set. Then $|A| < |\mathcal{P}(A)|$.

Remark Let A, B be sets. Exactly one of the following holds. $|A| = |B|$, $|A| < |B|$, $|A| > |B|$.

The Schröder-Bernstein Theorem Let A, B be sets. If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof of Schröder-Bernstein Theorem

* It is about making rooms for hotel guests.

We need the following the lemma proved in your homework.

Lemma If $f : R \to S$ and $g : T \to U$ are bijective (injective, surjective) and $R \cap T = \emptyset = S \cap U$, then $h : R \cup T \to S \cup U$ defined by $h(x) = \begin{cases} f(x) & \text{if } x \in R, \\ g(x) & \text{if } x \in T, \end{cases}$ is bijective (injective, surjective).