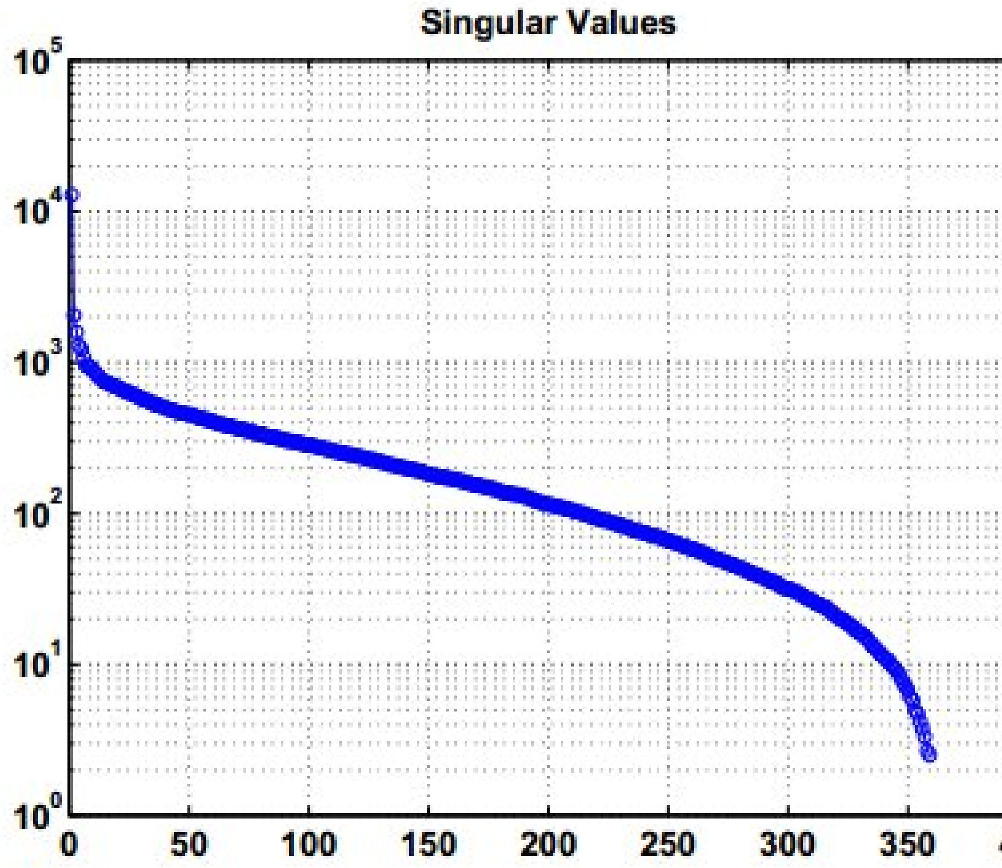
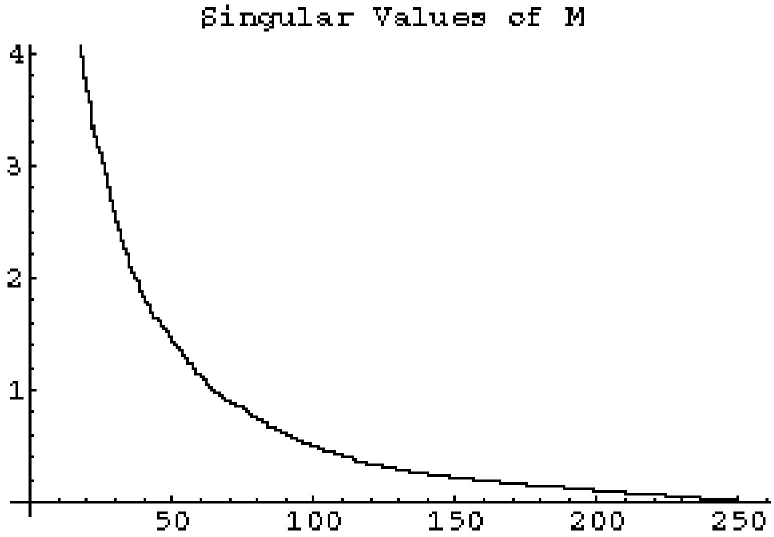
Q1:Investigation of the kind of images for which low rank SVD approximations can give very good results

General distribution of singular values with increase of k values:



(image taken from:http://www.columbia.edu/itc/applied/e3101/SVD_applications.pdf)



Quality of image is mostly measured either by:

1.(sum of singular value kept)/(original sum of singular value)

2.Frobenius norm of approximated image/Frobenius norm of original image.

Through experiments, I propose 3 criteria that the low rank SVD approximations can give good result by listing some examples:

1. The difference(euclidean norm) between two columns/rows are small compared to the Fronbenius norm of the matrix.

2. The difference(euclidean norm) between columns/rows with zero colums/rows is small compared to the Fronbenius norm of the matrix.

3. The columns/rows are near that another columns/rows in term of the cosine angle. The cosine is near 1.

In these cases, the difference between the Frobenius norm is small. While the best rank r-1 approximation to rank r matrix has "Frobenius norm difference" root of square of the rth singular values, it can be said that the rth singular value is small compared to the total root of sum of square. (2 rows/columns identical or with 1 more zero rows/columns or 2 rows/columns multiple of each other mean rank r-1 matrix)

Examples:

Classic example where low rank approximation is bad:

0.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.(The difference(euclidean norm) between two columns/rows are small compared to the Fronbenius norm of the matrix.)

```
-->A=[99 88 18;46 77 999;90 85 20]
 A  =


    99.     88.     18.
    46.     77.     999.
    90.     85.     20.

-->svd(A)
 ans  =


    1003.9386
    178.15745
    2.6689437
```

It can be seen that the Frobenius norm of difference of row 1 and row 3 is small compared to the Frobenius norm of matrix. Last singular value or even the second 1 can be truncated.

2.(The difference(euclidean norm) between columns/rows with zero colums/rows is small compared to the Fronbenius norm of the matrix.)

```
-->B=[10 0 0;0 10 0;0 0 1]
 B  =


    10.     0.      0.
    0.      10.     0.
    0.      0.      1.

-->svd(B)
 ans  =


    10.
    10.
    1.
```

It can be seen that the Frobenius norm of difference of row 3 and zero row is small compared to the Frobenius norm of matrix. Last singular value can be truncated.

3.(The columns/rows are near that another columns/rows in term of the cosine angle. The cosine is near 1.)

```
-->C=[1 100 11;23 150 15;100 200 20]
 C  =


    1.        100.      11.
    23.       150.      15.
    100.      200.      20.

-->svd(C)
 ans   =


    284.9111
    51.001483
    0.7157167
```

The cosine of 2nd and 3rd columns are close to 1. Last singular value is near 0.

4.(not obvious example)

```
-->D=[99 108 102;36 192 88;36 100 57]
 D  =


    99.     108.     102.
    36.     192.     88.
    36.     100.     57.

-->svd(D)
 ans  =


    295.65575
    70.040111
    0.243387
```

This example is a not so obvious where 3×column3−2×column1 nearly equal column2.

Last but not least:

5×5 example:

```
-->T=[1 99 200 89 67;0 50 100 63 12;2 90 181 77 13;2 120 241 11
 T  =

    1.     99.      200.     89.     67.
    0.     50.      100.     63.     12.
    2.     90.      181.     77.     13.
    2.     120.     241.     11.     2.
    1.     99.      199.     6.      1.

-->svd(T)
 ans  =

    486.82801
    98.200659
    34.523036
    1.1229673
    0.1487234
```

Therefore sometimes complex image may be well approximated by low-rank matrix.

I believe it is related to, but not implied by, the Eckart-Young theorem:

"In the case that the approximation is based on minimizing the Frobenius norm of the difference between A and which has a specific rank r. In the case that the approximation is based on minimizing the Frobenius norm of the difference between A and A' under the constraint that it turns out that the solution is given by the SVD of M, namely $A'=UX'V^{\star}$,where X' is the same matrix as X except it contains only the r largest singular values."

(wiki: singular value decomposition)

Another indicator of whether a truncation of the last singular value is good estimate can be done by calculating the condition number.

Condition number:

$\mathrm{cond}(A) = ||A||/||A^{-1}||$=largest singular value/lowest singular value.