Feb 15, 2012 Summary

Method: Obviously, when the transition matrix is diagonalizable, we can always use partial fraction decomposition to determine power series expansion of $(I-zP)^{-1}$. In general, we write $UPU^{-1}=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$. Observed that if the first row of U is normalized to π, then the first column of U^{-1} must be normalized to **1** since $UU^{-1}=I$ and hence $(UV)_{11}=u_1v_1=\pi v_1=1$. (Markov Chain and Random Walks on Graph)

Denote $\Lambda=\begin{pmatrix}1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}$, $P^t= U^{-1}\begin{pmatrix}1 & & & \\ & \lambda_2^t & & \\ & & \ddots & \\ & & & \lambda_2^t\end{pmatrix}U$

$\xrightarrow{t\to\infty}\begin{pmatrix}v_{11}u_1\\ v_{12}u_1\\ \vdots\\ v_{1n}u_1\end{pmatrix}=\begin{pmatrix}\pi\\ \pi\\ \vdots\\ \pi\end{pmatrix}$, since P has the unique largest eigenvalue $\lambda_1=1$ by Perron-Frobenius Theorem. One may consider $1\geq\neq|\lambda_2|\geq|\lambda_3|\geq\cdots\geq|\lambda_n|$, generally, given an initial distribution q_0, $q^{(t)}=qP^t=\pi+\sum_{i=2}^{n}q_i^{'}\lambda_i^t u_i$, where $q_i^{'}=\frac{qu_i^T}{\|u_i\|^2}$, u_i's are just the eigenvalue basis for P.

Numerical example:

Diagonalize $P=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}$, $\begin{pmatrix}2 & 1\\ -3 & 1\end{pmatrix}^{-1}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{3}{4} & \frac{1}{4}\end{pmatrix}\begin{pmatrix}2 & 1\\ -3 & 1\end{pmatrix}=\begin{pmatrix}-\frac{1}{4} & 0\\ 0 & 1\end{pmatrix}$

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} (-\frac{1}{4})^n & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

$$(I - zP)^{-1} = (1 + z + z^2 + z^3 + \cdots)\begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{pmatrix} + [1 - \frac{z}{4} + (\frac{z}{4})^2 - (\frac{z}{4})^3 + \cdots]\begin{pmatrix} \frac{2}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{3}{5} \end{pmatrix}$$

<u>Suggestion:</u> Consider the following three schemes.

1. Simply write $P = \sum_{i=1}^{n} \lambda_i u_i v_i$, where u_i, v_i's are the column and row vectors of U^{-1} and U respectively.
2. Use the method like this: calculate $(I\text{-}zP)^{-1}$ and write it in the form of $\sum_{j=1}^{n}\left(\sum_{i=1}^{n}\gamma_i z^i\right)B_j$.
3. Study Cayley-Hamilton Theorem, and try to figure out if we can represent the power of *P* in terms of some lower degree matrix polynomials.