Power of Matrices.

① First of all, if we know the matrix P is diagonalizable.

Let $P = S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} S^{-1}$ write $S = (\underset{\sim}{x_1}, \underset{\sim}{x_2}, \cdots, \underset{\sim}{x_n})$. $S^{-1} = \begin{pmatrix} y_1^T \\ y_2^T \\ \vdots \\ y_n^T \end{pmatrix}$

Then $P = \lambda_1 \underset{\sim}{x_1} \underset{\sim}{y_1}^T + \lambda_2 \underset{\sim}{x_2} \underset{\sim}{y_2}^T + \cdots + \lambda_n \underset{\sim}{x_n} \underset{\sim}{y_n}^T$

and $P^m = \lambda_1^m \underset{\sim}{x_1} \underset{\sim}{y_1}^T + \lambda_2^m \underset{\sim}{x_2} \underset{\sim}{y_2}^T + \cdots + \lambda_n^m \underset{\sim}{x_n} \underset{\sim}{y_n}^T$

② denote $\det(xI - A) = p(x)$.

By Cayley-Hamilton Theorem, $P(A) = O_n$

Then $A^m = P(A)\, q(A) + r(A) = r_0 I_n + r_1 A + \cdots + r_{n-1} A^{n-1}$

And $A^m = r_0 I_n + r_1 S \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} S^{-1} + r_2 S \begin{pmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_n^2 \end{pmatrix} S^{-1} + \cdots + r_{n-1} S \begin{pmatrix} \lambda_1^{n-1} & & \\ & \ddots & \\ & & \lambda_n^{n-1} \end{pmatrix} S^{-1}$

$= S \left(r_0 I_n + r_1 \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} + \cdots + r_{n-1} \begin{pmatrix} \lambda_1^{n-1} & & \\ & \ddots & \\ & & \lambda_n^{n-1} \end{pmatrix} \right) S^{-1} = S \begin{pmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{pmatrix} S^{-1}$

$$\Longleftrightarrow \begin{bmatrix} \lambda_1^m \\ \vdots \\ \lambda_n^m \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix}$$

③ We use partial fraction decomposition to write $(I - zA)^{-1}$ in power series, and compare the coefficients to get A^m.

$$(I - zA)^{-1} = \frac{1}{\det(I - zA)} \operatorname{adj}(I - zA) = \left(\sum_{k=0}^{\infty} \gamma_k z^k\right)\left(I + B_1 z + \cdots + B_{n-1} z^{n-1}\right)$$

$= \sum_{m=0}^{\infty} z^m A^m$ Then $S \begin{bmatrix} \lambda_1^m & & \\ & \ddots & \\ & & \lambda_n^m \end{bmatrix} S^{-1} = \gamma_m I + \gamma_{m-1} B_1 + \cdots + \gamma_{m-n+1} B_{n-1}$

Numerical Example.

```
>> P=[
0.5 0.4 0.1
0.3 0.4 0.3
0.2 0.3 0.5]
>> d=eig(P)
d =
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$\lambda_1 =$ 1.0000

$\lambda_2 =$ 0.3414

$\lambda_3 =$ 0.0586

```
>> [S,D]=eig(P)
S =
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$$x_1 = \begin{pmatrix} -0.5774 \\ -0.5774 \\ -0.5774 \end{pmatrix} \quad x_2 = \begin{pmatrix} -0.7135 \\ 0.1098 \\ 0.6920 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0.6261 \\ -0.7469 \\ 0.2239 \end{pmatrix}$$

```
D =
    1.0000         0         0
         0    0.3414         0
         0         0    0.0586
>> S^(-1)=inv(S)
S^(-1) =
```

$y_1 = ($ -0.5867 -0.6425 -0.5029 $)$

$y_2 = ($ -0.6073 -0.2516 0.8589 $)$

$y_3 = ($ 0.3642 -0.8792 0.5150 $)$

① $P^m = x_1 y_1 + \lambda_2^m x_2 y_2 + \lambda_3^m x_3 y_3$

Stationary distribution $\pi = \left(\frac{21}{62} \; \frac{23}{62} \; \frac{18}{62} \right)$

② $P^m = r_0 I_3 + r_1 P + r_2 P^2 + \cancel{r_3}$

where

$$\begin{pmatrix} 1 \\ 0.3414^m \\ 0.0586^m \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0.3414 & 0.3414^2 \\ 1 & 0.0586 & 0.0586^2 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \end{pmatrix}$$

③ $$(I - zP)^{-1} = S \begin{pmatrix} \frac{1}{1-z} & & \\ & \frac{1}{1-0.3414z} & \\ & & \frac{1}{1-0.0586z} \end{pmatrix} S^{-1}$$

$$= S \begin{pmatrix} 1+z+z^2+\cdots & & \\ & 1+(0.3414z)+(0.3414z)^2+\cdots & \\ & & 1+(+0.0586z)+(0.0586z)^2+\cdots \end{pmatrix} S^{-1}$$

$$(I - zP)^{-1} = \frac{\operatorname{adj}(I - zP)}{\det(I - zP)}$$

In practice, to calculate the adjoint of $(I - zP)$ is too complicated.

And $\det(I - zP)$ is analytic, we can write it as power series of z.