From Galileo to Supersymmetry: Algebra in Physics

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Abstract

Since Galileo established the fundamental principles for modern sciences, physics has undergone several revolutions and the framework of the subjec, as well as our understanding of nature, has improved significantly. As mathematics has always been the language by physics laws are formulated and articulated, many of its branches all have their unique and profound applications in different aspects of physics. In this writing I will be talking mostly about the application of abstract algebra in physics, more specifically the various groups through which physics laws are expressed. I intend to show that algebra has accompanied physics laws from the very foundation of the subject, to the most advanced frontier of Physics Beyond Standard Model, and the underlying similarities they are share and the reason why the evolution of physics laws is consistent.

Contents

1 Introduction

Galileo postulated his free fall formula in the 1600s, which laid the foundation for modern physics. Since then, numerous contribution were made by famous names like Issac Newton, and the ideas such as Galilean Transformation dominated the way people thought about the laws of nature at that time. Later it was chanllenged by Einstein in the early 1900s that Galilean Transformation is no longer accurate under the extreme conditions of very high speed thus the formulation of Lorentz Group. It was also noticed that some new sets of rules were needed when dealing with extremely small objects and hence the quantum mechanical operations. Eventually in an effort to combine special relativity and quantum mechanics, the Standard Model was created and a more refined version of interpretation was made available. Now as people look into the limitations of the Standard Model they are postulating new ideas, generally called Physics Beyond Standard Model in order to yet improved our understanding of the world. Most part of the story was no stranger to many people, but the underlying mathematical connections between these seemingly evolving and somewhat seperate theories are much stronger than their deceiving looks as simply many formulas. Indeed it is the group structures, Galilean Groups, Lorentz Groups, Standard Model Representations, and Supersymmetry algebra, behind these progress that are actually highly consistent and reveal much about the fundamental properties of modern physics through purely their mathematical forms.

2 Galilean Transformation and Galilean Groups

In a most natural sense, if we imagine us in a car passing a resting object the intuitive observation should be that the object is seemingly moving backward, but besides that nothing in the two reference frames (ours and the resting reference frame) should differ. In fact that was what Newton imagined as well. He established the idea of absolute spacetime, which insists that time is seperate entity from space and is not affected by motion whatsoever.

The ideas then led to the so called Galilean transformation:

$$x' = x - vt, y' = y, z' = z, t' = t \tag{1}$$

Where x, y, z are the natural coordinates for 3 spatial dimensions and t denotes time, and the primed letter indicates the same event observed in the moving reference frame. Obviously this agrees with our every observation and therefore was the basic idea for physics for almost 300 years before being challenged.

Now after seeing the easy physics part, let us think more about the mathematical structure it might induce. The logic is that if there are two observers A and B and traveling along two directions, the combined results should simply equivalent to a "new" observer traveling on the combined direction considering the speed, much like adding two forces in vector addition. In another words, Galilean Transformations are closed. Thus we postulate that the Galilean Transformations form a group in itself called the Galilean Group.

From the newest research, [1], it is suggested that the Galilean Group $SGal(3)$ (in 3 spatial dimensions) takes the form

$$(t, x, 1) \begin{pmatrix} 1 & v & 0 \\ 0 & R & 0 \\ s & y & 1 \end{pmatrix} \tag{2}$$

Where $(t, x, 1)$ is an arbitrary event, with x and y both being a 3-dimensional vector, denoting the position vector of the object and the displacement of the reference frame, respectively. Now v being velocity, s as elapsed time, R is the rotation matrix and $R \simeq SO(3)$, y is the other coordinate. Since all the parameters are real number, this is an example of a continuous group (which will be mostly the case in physics). The Galilean Group is also a Lie Group defined by some bracket operations, which shows the symmetry embedded in the group structure.

Now let's consider the outcome of such operation. The result of the above representation is $(t + s, tv + xR + y, 1)$. Comparing with (1), we see that it is indeed a slightly more complicated transform involving multiple aspect. The time is shifted by s, and instinctively a displacement due to motion vt, as well as a rotation R on the position of the object and eventually a shift of origin y.

Now following this most generic construction, we can ask ourselves: are there any subgroups to this Galilean Group? Obviously this is a continuous group with uncountable order, but a natural guess would prompt that we reduce the factors involved in (2) and try to see if different parts of the operations would form groups in themselves. And indeed it is the case here.

Because R denotes rotation so if we take out everything else we should get a group containing structure isomorphic $SO(3)$. The reason being that the matrix representation of rotation, for example a rotation of angle ϕ around z axis:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

Form a group of determinant 1 and are all mutually orthogonal. And if we simply treat the diagonal entries to be 1 and off-diagonal ones to be 0 in (2), then they form a group exactly the way of $SO(3)$ since the first and fifth entries do not interfere with R in multiplication.

The second group, on the other hand, sets R to I_3 and only the shift of origin y is considered. Instinctively two shifts of origin will simply combine and become a new shift of origin, and that's the logic in the subgroup G_2. Similarly, we can consider the case where we set our entire reference frame in motion with respect to the original frame, in which $y = 0$ and $s = 0$, it again result in a subgroup. And all the matrix multiplication can be easily verified.

Some addition insights: this follows from the mathematical formulation of the theory: we can see that s plays little role in Galilean Groups. This is reflected the separation of space and time, which

3 Lorentz Transform and Lorentz Group

The ideas of Galilean Transformation remained dominant for the next few hundred years. In the later 19th century, Maxwell developed his complete theory of electromagnetism and one of its direct consequences is that the speed of light is always the same in any reference frame. This then contradicts the idea of Galilean universe if we consider a light signal traveling between x_1 and y_1, when one of them is along the direction of motion, we would then end up with different speeds of light. Einstein proposed the resolution of this dilemma and introduces the special relativity which replaces Galilean Transformation with Lorentz Transformation [2]. When considering a boost in the x direction:

$$t' = \gamma(t - \frac{vx}{c^2}), x' = \gamma(x - vt), y' = y, z' = z \tag{4}$$

Where $\gamma = \frac{1}{\sqrt{1-\beta}}$ and $\beta = \frac{v}{c}$. The significance of this idea is that it combines space and time and turn them into one entity spacetime, that the sense of time is not absolute and just like position, depends on relative motion.

So how is this reflected in the representation? The standard way is the representation:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{5}$$

, in the case a of x boost. In the cases of y, z boosts, we preserve the rows and columns corresponding to t and permute the row and columns corresponding to x to y or z. Again, in the same logic of Galilean Groups, these operations are closed (two x boosts are still x boosts). Keep in mind that the fundamental aspects of boost, rotations and shifts of origins are still present in Lorentz Transformation, only the way they manifested become somewhat more difficult.

If we try to set Galilean Transformations in this manner,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{6}$$

This does not reveal much about the essence of Galilean Transformations and that is why we much prefer the previously mentioned representation of Galilean Group. And notice the very keen difference in this formulation, the off-diagonal elements are no longer symmetric in Galilean Transformation. This is rather fundamental about the difference between the two theories, as it is another evidence of the spacetime symmetry resembled by the Lorentz Transformation and the isolated nature of time in Galilean formulation. That is also the most fundamental symmetries contained in Lorentz Group and why the Group is so important.

In comparison with the Galilean Group, what has changed in the context of Lorentz Groups? Apparently the operations remains the same, but the relations between t and other variables changed.

Looking into the matrix representation and recall the Galilean Group notation:

$$(t, x, 1) \begin{pmatrix} 1 & v & 0 \\ 0 & R & 0 \\ s & y & 1 \end{pmatrix} = (t + s, tv + xR + y, 1) \tag{7}$$

Does the similar representation exist in Lorentz Group? The fact that time is combined with the notation of space should render this difficult. We might be able to try the following operations that transform t and x according to the form of Lorentz Transformation:

$$(t,x,1)\begin{pmatrix} \gamma & \gamma v & 0 \\ \gamma(\frac{v}{c^2}) & \gamma R & 0 \\ s & y & 1 \end{pmatrix} = (\gamma t + \gamma(\frac{vx}{c^2}) + s, \gamma(vt + xR) + y, 1) \tag{8}$$

So we still might be able to write the group under the transformation, and the $v = 0$ case should constitute a subgroup. But we see some of the notations, such as s should change according to special relativity as well. On the other hand, if $v \neq 0$ then $\gamma > 1$ always. We see that the operation gives

$$\begin{aligned} &\begin{pmatrix} \gamma_1 & \gamma_1 v_1 & 0 \\ \gamma_1(\frac{v_1}{c^2}) & \gamma_1 R_1 & 0 \\ s_1 & y_1 & 1 \end{pmatrix}\begin{pmatrix} \gamma_2 & \gamma_2 v_2 & 0 \\ \gamma_2(\frac{v_2}{c^2}) & \gamma_2 R_2 & 0 \\ s_2 & y_2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1\gamma_2 + \gamma_1\gamma_2(\frac{v_1 v_2}{c^2}) & \gamma_1\gamma_2 v_2 + \gamma_1\gamma_2 v_1 R_2 & 0 \\ \gamma_1\gamma_2(\frac{v_1}{c^2}) + \gamma_1\gamma_2(\frac{v_1 R_2}{c^2} & \gamma_1\gamma_2(\frac{v_1 v_2}{c^2}) + \gamma_1\gamma_2 R_1 R_2 & 0 \\ s_2 & y_2 & 1 \end{pmatrix} \end{aligned} \tag{9}$$

We see that the boost is always larger than the values that the two boosts combined regardless of the direction. This problem does not exist in Galilean representation as the speeds obey simple addition rules. So this representation is not as viable as (5).

3.1 Generalization of Lorentz Group

Now let's talk a little bit more about the representation of Lorentz Group. Incidentally equation (5) might not always be the correct representation of a Lorentz Group, as we are considering an orthogonal sets of basis vectors completely aligned with x, y, z, t axis. Naively one would think of applying a change of basis unitary matrix U to the transformation matrix G, so that $G' = U^\dagger G U$ should preserve the matrix form of the Lorentz Transformations. Yet more, the matrix in (5) can be generalized even further if we are interested in not only strictly the physical Lorentz Transformation. For example the parabolic Lorentz Group can be generalized as the form:

$$\begin{pmatrix} 1+\phi^2/2 & \phi & 0 & -\phi^2/2 \\ \phi & 1 & 0 & -\phi \\ 0 & 0 & 1 & 0 \\ \phi^2/2 & \phi & 0 & -\phi^2/2 \end{pmatrix} \tag{10}$$

The parameters are determined by the exact form of operations acting on the base vector field, which is not necessarily physical.

It is worth pointing out that a very important result of special relativity is the idea of proper time, that is $ds^2 = cdt^2 - dx^2 - dy^2 - dz^2$ which is the measure of "distance of spacetime" between events. Lorentz Transformation preserves this proper time between events in the presence of Lorentz boosts. So mathematically, the alternative representation is that any group isomorphic to $O(1,3)$ that preserves the quantity $s^2 = t^2 - x^2 - y^2 - z^2$. This leaves Lorentz Group much more powerful than Galilean Group and thus can be applied to much more aspects of physics again.

3.2 Extension of Lorentz Group

We have seen that the natural setting for Lorentz Group is in $3+1$ dimensions. But it is reasonable to generalize it to the case of $SO(1,n)$, where there are more than 3 spatial dimensions so that it incorporates the ideas of string theory and M-theory, to show the existence of extra dimensions and still remain consistent.

In another words, equation (5) can be generalized into:

$$\begin{pmatrix} ct' \\ x'_1 \\ x'_2 \\ \cdots \\ x'_n \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & \cdots & 0 \\ -\beta\gamma & \gamma & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} ct \\ x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \tag{11}$$

This is easily done mathematically, as the structure of the matrix should not change under the dimension change from $3+1$ to $n+1$. Again, relating to our previous discussion of the matrix form of the operation, we might be thinking of the same idea of a n-dimensional, instead 4-dimensional, change of basis matrix U applied to the transformation matrix would give a consistent representation of the matrix under any basis, in other words, we consider $G' = U^\dagger G U$. Of course the more generic form of the group representation will be any isometry that preserves

$$s^2 = t^2 - x_1^2 - \cdots - x_n^2 \tag{12}$$

Since in natural units we are allowed to set $c = 1$.

If we want to consider the physical significance of the theory, since our observation shows that there are only 3 macroscopic spatial dimensions that are detectable so far, people has speculated that the other dimensions take a different form and cannot be thought as identical to the x, y, z spatial dimensions that we are used to. From some ideas of Edward Witten [4], we might be able to imagine the structure of the spacetime as $SO(1,3) \times S^7$ where S^7 is a topological group representing the 7 extra dimensions. This is a beautiful way of relating the conjecture that there are other hidden microscopic dimensions in potential physics models.

We can also combine translation with Lorentz Transformation to get Poincare Group but that is the topic for later discussion. To summarize, Lorentz Groups only concern with

Lorentz transformation. Again this group is of utter most importance because it contains the symmetry operation that are invariant under Lorentz Transformation. The natural setup for the Lorentz Group is in the $3+1$ dimensions, where the Group is isomorphic to $SO(1,3)$.

4 Poincare Group and spacetime symmetries

4.1 Schrodinger's equation and Spacetime transforms

We then must ask the question that how are these ideas useful in promoting our understanding of other aspects of physics? One of the examples would be in the case of Schrodinger's equation, which takes the form:

$$H\psi(x) = E\psi(x) \tag{13}$$

Where H is the Hamiltonian operation, usually taking the form $\frac{h^2}{2m}\frac{d^2}{dx^2} + V(x)$, representing both the kinetic energy and the potential energy V. E is the energy eigenvalues and ψ are eigenstates of the system. Usually this equation only allows certain discrete number of states and because the operator is Hermitian these eigenstates form orthogonal basis. The Schrodinger's equation is only invariant under Galilean Transformation, as it does not contain Lorentz invariance terms in the time-independent form of (**??**):

$$H\phi(x,t) = ih\frac{\partial}{\partial t}\phi(x,t) \tag{14}$$

So people realize the formation is not complete [3]. Then ideas were brought up such as the Klein-Gordon equation, which adds terms in the form of $\frac{d^2}{dt^2}$ to the schrodinger's equation so that the formulation is Lorentz-Invariant. This eventually brought forth the Standard Model.

4.2 The Standard Model

Now that we can extend the Lorentz Group to get a larger group which absorbs all the isometry operations in flat spacetime, or Minkowski Space, including translational, rotational, and boost symmetry. The Poincare Group combines translation and Lorentz Transformation together and it is used in representation of Standard Model algebra.

After Einstein's theories of relativity, people had been looking for more ways to incorporate particle theory and special relativity. SO far the Standard Model has been the most reliable and comprehensive way we interpret physics. The Standard Model incorporates the three fundamental interactions, electromagnetic, strong, and weak interactions into one consistent picture. Here the same symmetry group ideas still applies, since Lorentz Invariance is a basic requirements for the particles in the Standard Model, that is, the spacetime property of particles should not change under reference frame changes.

In many situations particular kinds of particles would preserve their properties, or convert into each other but never into other particles. We classify them all under specific symmetry groups. The reasons being that the symmetry groups contain the transformations of spacetime, or rather the choice of different observers in the context of physics, which should not change the laws of physics. Therefore we can characterize our theory by these symmetry groups. For example, the mathematical foundation of the Standard Model can be characterized as $SU(3) \times SU(2) \times U(1)$, for which different kinds of elementary particles would interaction with various parts of the symmetry group.

If we consider quarks, we can denote that up quark to be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and down quarks to be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. They are then connected by a unitary matrix A where $A \simeq SU(2)$. Another example would be electroweak which is thought to be interacting with the $U(1) \times SU(2)$ sector, and the quarks and gluons interactions are characterized by $SU(3)$ symmetry. Notice that these operations are all invariant under Lorentz Transformation.

5 Supersymmetry Algebra

The Standard Model is not the complete picture because it fails to unify gravitational interaction with the other three, along with other various unsolved problems. People have since proposed models to further improve understanding of nature based on the Standard Model, which are generally called Physics Beyond the Standard Model. One of the most popular models is the Supersymmetry idea, which proposed that beyond the 60 fundamental particles, there are ”superpartners” of each particle and it relates the ferminions with the bosons, which are half integer spin particles and integer spin ones, respectively. Naturally we construct groups to describe the different kinds of particles in the Standard Model, based on various symmetries. Then interestingly, one of the fundamental aspects about the mathematics of Supersymmetry is that its algebra can be seen as the extension of Poincare group, called the Super-Poincare Groups. The details are rather complicated and will not be the focus of discussion. THe important point is that these ideas are united under the developing, yet consistent algebra framework, and indeed why the model can be theoretically viable.

6 Conclusion

In this writing I surveyed various theories involving groups, from the most basic Galilean Groups to the very sophisticated Super-Poincare Group. Although the theories have been evolving fundamentally, we can still find the underlying mathematical structure to be related to each other and highly relevant. The inherent consistency between the theories are also illustrated and carried over by the mathematical descriptions. Algebra in this case, provides a universal way to connect different physics models together and enable us to make further progress based on conjecture of the possible extension of these structure and relate them to real life phenomenon.

References

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