Math 307 Abstract Algebra Examination 1 Name Sample Solution.

1. Let $*$ be an associative binary operation on a set G satisfying
   (a) there is a left identity $e \in G$ such that $e * a = a$ for all $a \in G$, and
   (b) for every $a \in G$, there is a left inverse a' such that $a' * a = e$.

   Show that the left inverse a' also satisfies $a * a' = e$, and the left identity also satisfies $a * e = a$. [Hint: Show that the left identity $(a')'$ of a' equals a.]

Note $g' = eg' = g'gg'$.

Operating $(g')'$ on both sides.

$$e = (g')'g' = (g')'(g'gg') = gg'.$$

So $gg' = e$ also.

Now $ge = g(g'g) = eg = g$ because $gg' = e$.

$\therefore$ $ge = g$.

2. Let H and K be subgroups of a group G. Show that $H \cup K \le G$ if and only if $H \le K$ or $K \le H$.

If $H \le K$ then $H \cup K = K \le G$

If $K \le H$ then $H \cup K = H \le G$.

Conversely. If $H \not\le K$, $K \not\le H$, then there ~~is~~ are $h \in H \setminus K$ and $k \in K - H$.

Assume the contrary that $H \cup K$ is a subgroup.

Then $h, k \in H \cup K \Rightarrow hk \in H \cup K$.

$\Rightarrow hk \in H$ or $hk \in K$.

If $hk \in H$, then $\underbrace{h^{-1}(hk)}_{=k} \in H$!!!

If $hk \in K$, then $h = (hk)k^{-1} \in K$!!!

3. Let G be a group. Show that $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is an isomorphism if and only if G is Abelian.

Assume $\phi(g)=g^{-1}$ defines an isom.

Then $(xy)^{-1}=\phi(xy) = \phi(x)\phi(y) = x^{-1}y^{-1}$ for any $x,y\in G$.

So $\cancel{(xy)(xy)^{-1}(xy) = x}$ $y^{-1}x^{-1} = x^{-1}y^{-1}$ $\forall x,y \in G$

Hence $\cancel{xy = xy(y^{-1}x^{-1})(xy)}$ $e = (xy)(y^{-1}x^{-1}) = (xy)(x^{-1}y^{-1})$

$\therefore yx = (xy)(x^{-1}y^{-1})yx = xy.$

So G is Abelian

Conversely, Suppose G is Abelian.

Define $\phi: G\to G$ by $\phi(g)=g^{-1}$.

$\forall x,y\in G$: $\phi(xy)=(xy)^{-1}=y^{-1}x^{-1}=x^{-1}y^{-1}$
$=\phi(x)\phi(y)$

One-one: $\phi(x)=\phi(y)$
$\Rightarrow x^{-1}=y^{-1} \Rightarrow x=y$. $\therefore$ 1-1

Onto: Let $y\in G$. Set $x=y^{-1}$

Then $\phi(x)=(y^{-1})^{-1}=y$.

$\therefore \phi: G\to G$ is an isomorphism.

4. Let G be a group with $|G| = pq$, where p, q are primes. Prove that every proper subgroup of G is cyclic. But the whole group may not be cyclic.

[Hint for the last part: Consider a group with 6 elements.]

Suppose $H \le G$. $H \neq G$.

Then $|H|$ is a factor of $|G|$

So $|H| = p, q, 1$.

Hence H is cyclic.

Consider $G = S_3$. Then G is not cyclic

with $\underset{2\cdot 3}{\underset{\parallel}{6}}$ elements

5. Let $H \leq S_n$. Show that either $H \leq A_n$ or $|H \cap A_n| = |H|/2$.

Case 1. $H \leq A_n$ we are done

Case 2: $H - A_n$ has an element σ which is an odd permutation.

Define $f: H \cap A_n \to H - A_n$ by

$$f(\pi) = \sigma \circ \pi.$$

① Well-defined: $\pi \in H \cap A_n$ is an even permutation. $\sigma \in H - A_n$ is odd. $\therefore$ $\sigma\pi \in H$ is an odd permutation. So $\sigma H \in H - A_n$.

② 1-1: $f(\pi_1) = f(\pi_2) \Rightarrow \sigma\pi_1 = \sigma\pi_2 \Rightarrow \pi_1 = \pi_2$ by cancellation

③ Onto: If $\tau \in H - A_n$ then let $\pi = \sigma^{-1}\tau$. $\sigma^{-1} \in H$, $\tau \in H \Rightarrow \sigma^{-1}\tau \in H$. Also σ^{-1} is odd, τ is even $\Rightarrow$ $\sigma^{-1}\tau$ is even. So $\pi \in H \cap A_n$. Now $f(\pi) = \tau$.

Hence f is bijective & $|H \cap A_n| = |H - A_n|$. We see that $|H \cap A_n| = \frac{|H|}{2}$.

6. Find non-trivial proper subgroups H_1, H_2 of S_3 such that

(a) $\sigma H_1 = H_1 \sigma$ for all $\sigma \in S_3$, and (b) $\sigma H_2 \neq H_2 \sigma$ for some $\sigma \in S_3$.

List all the cosets of H_1 and H_2 to illustrate your answers.

(a) Let $H_1 = \langle (1,2,3) \rangle = \{(1,2,3), (1,3,2), \varepsilon\}$ (ε: Identity).

Then $(1,2) H_1 = \{(1,2), (1,3), (2,3)\} = H_1 (1,2)$ is the only other coset. So for any $\sigma \in S_3$, $\sigma H_1 = H_1 = H_1 \sigma$ or $\sigma H_1 = (1,2) H_1 = H_1 \sigma$.

(b) Let $H_2 = \langle (1\,2) \rangle = \{(1,2), \varepsilon\}$

$(1,3) H_2 = \{(1,3), (1,2,3)\}$
$(2,3) H_2 = \{(2,3), (1,3,2)\}$ are the two other left cosets.

$H_2 (1,3) = \{(1,3), (1,3,2)\}$
and $H_2 (2,3) = \{(2,3), (1,2,3)\}$ are the two other right cosets

Evidently, $(1,3) H_2 \neq H_2 (1,3)$, $(2,3) H_2 \neq H_2 (2,3)$.

Question	1	2	3	4	5	6	Total
Score							