MATH 307 Final Examination (Take home component). Due: May 8, Monday 5:00 p.m.

1. Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be a ring homomorphism. **Prove or disprove** the following.

   (a) If A is an ideal of R_1, then $\phi(A)$ is an ideal in R_2.

   (b) If B is an ideal of R_2, then $\phi^{-1}(B) = \{a \in R_1 : \phi(a) \in B\}$ is an ideal of R_1.

2. Let $\mathbf{D}$ be an integral domain with unity 1.

   (a) Show that $\tilde{\mathbf{D}} = \{r \cdot 1 : r \in \mathbf{Z}\}$ is a subdomain of $\mathbf{D}$, **and** $\tilde{\mathbf{D}}$ is contained in every subdomain of $\mathbf{D}$. (Make sure that that subdomain has same unity.)

   (b) Show that the characteristic of any subdomain of $\mathbf{D}$ is the same as that of $\mathbf{D}$.

3. Let $\mathbf{F}$ be a fields. Suppose $f(x) \in \mathbf{F}[x]$, and $A = \langle f(x) \rangle = \{f(x)h(x) : h(x) \in \mathbf{F}[x]\}$.

   (a) If $f(x) \in \mathbf{F}[x]$ is reducible[1], show that the factor ring $\mathbf{F}[x]/A$ is **not** an integral domain. (It follows that $\mathbf{F}[x]/A$ is not a field.)

   (b) Suppose $f(x)$ is irreducible. Show that A is an maximal ideal. (It follows that $\mathbf{F}[x]/A$ is a field.)

   [Hint: If A is not maximal, then there is an ideal B in $\mathbf{F}[x]$ not equal to A or $\mathbf{F}[x]$ such that $A \subset B \subset \mathbf{F}[x]$. Suppose $g(x) = g_0 + \cdots + g_m x^m$ with $g_m = 1$ is a polynomial in B with **minimum** degree. Show that $m > 1$ and $f(x) = g(x)h(x)$ such that $h(x)$ has degree larger than 1 by the division algorithm Theorem 16.2.]

4. Let $A = \langle x^2 + 1 \rangle \subseteq \mathbf{Z}_3[x]$, and $\mathbf{E} = \mathbf{Z}_3[x]/A$.

   (a) Show that $f(x) = x^2 + 1$ is irreducible. [Only need to show that $f(a) \neq 0$ for all $a \in \mathbf{Z}_3$.]

   (b) Determine (with proof) all the generators of the cyclic group $(\mathbf{E}^*, \dot{})$.

   [Hint: $\mathbf{E}^*$ is isomorphic to $\mathbf{Z}_8$, and should have 4 generators, each has order 8.]

   (c) Find the inverse of $2x + 1 + A$ in $\mathbf{E}$.

Good Luck!

[1] That is, $f(x) = f_1(x)f_2(x)$ such that $f_1(x), f_2(x)$ has degrees strictly smaller than that of $f(x)$.