Math 307 – Abstract Algebra **Homework 1** Sample solution

1. Let a and b be integers. Show that $a \mod n = b \mod n$ if and only if $a - b$ is divisible by n.

   Solution. Suppose $a \mod n = b \mod n$. Then $a = nq_1 + r$ and $b = nq_2 + r$ for some $r \in \{0, 1, \ldots, n-1\}$. Thus, $a - b = nq_1 - nq_2 = n(q_1 - q_2)$ is a multiple of n.

   Conversely, if $a - b$ is divisible by n, then $a - b = nq$. If $a = nq_1 + r$ with $r \in \{0, 1, \ldots, n-1\}$, then $b = a - nq = nq_1 + r - nq = n(q_1 - q) + r$. Thus, $a \mod n = b \mod n$.

2. Show that $5n + 3$ and $7n + 4$ are relatively prime for all $n \in \mathbb{N}$.

   Solution. Note that $a, b \in \mathbb{Z}$ are relatively prime if and only if there is $x, y \in \mathbb{Z}$ such that $ax + by = 1$.

   Using Euclidean Algorithm, we have $3(5n + 3) - 2(7n + 4) = 1$. So, $\gcd(5n + 3, 7n + 4) = 1$.

3. Determine $7^{1000} \mod 6$, and $6^{1001} \mod 7$.

   Solution. In $\mathbb{Z}_6$, $7^{1000} = 1^{1000} = 1$; $6^{1001} = (-1)^{1001} = -1 = 6$.

4. Prove that $2^n 3^{2n} - 1$ is always divisible by 17 for all non-negative integers n.

   Solution. For $n = 0$, $2^n 3^{2n} - 1 = 0$ is divisible by 17.

   Assume $2^k 3^{2k} - 1 = 17m$ for some integer $k \geq 0$. Then $2^{k+1} 3^{2k+2} - 1 = (18) 2^k 3^{2k} - 1 = 18(2^k 3^{2k} - 1) + 17 = 17(18m + 1)$ is divisible by 17. By Principle of M.I., the conclusion holds.

5. Let $\mathbb{R}$ be the set of real numbers. Define a relation R on $\mathbb{R}$ by $(a, b) \in R$ if $a - b$ is an integer. Prove that R is an equivalence relation and determine the equivalence classes.

   Solution. We need to show that R is reflexive, symmetric, and transitive.

   Reflexive. If $a \in \mathbb{R}$, then $(a, a) \in R$ because $a - a = 0$ is an integer.

   Symmetric. If $(a, b) \in R$, then $a - b$ is an integer and so is $b - a$. Thus, $(b, a) \in R$.

   Transitive. If $(a, b), (b, c) \in R$, then $a - b = r, b - c = s$ are integers. Thus, $a - c = (a - b) + (b - c) = r + s$ is an integer. So, $(a, c) \in R$.

   To determine all the equivalence classes, note that the equivalence class for $r \in \mathbb{R}$ is $[r] = \{x \in \mathbb{R} : x - r \in \mathbb{Z}\} = \{r + k : k \in \mathbb{Z}\}$. We know that $[r] = [s]$ if $r - s$ is an integer. Hence, the set of all distinct equivalence classes are $[r]$ with $r \in [0, 1)$.

6. Express the following complex numbers in standard form:

   $$(-7 - 3i)^{-1}, \quad \frac{-5 + 2i}{4 - 5i}.$$

   Solution.

   $$(-7 - 3i)^{-1} = \frac{-7 + 3i}{(-7 - 3i)(-7 + 3i)} = \frac{-7}{58} + i\frac{3}{58}.$$

   $$\frac{-5 + 2i}{4 - 5i} = \frac{(-5 + 2i)(4 + 5i)}{(4 - 5i)(4 + 5i)} = \frac{-30}{41} + i\frac{-17}{41}.$$

7. (Extra credit) prove that none of the integers $11, 111, 1111, 11111, \ldots$ is a square of an integer. How about 2, 22, 222, 2222, ..., 3, 33, 333, 3333, ...; etc.?

   Solution. If $n \in \mathbb{Z}$ ends with $1, 2, 3, 4, 5, 6, 7, 8, 9$, then n^2 ends with $1, 4, 9, 6, 5, 6, 9, 4, 1$. So, n^2 cannot be of the form $2 \cdots 2$, $3 \cdots 3$, $6 \cdots 6$, $7 \cdots 7$, $8 \cdots 8$,

   Note that if $n^2 = 11, 11, 111, 1111, \ldots$, the last digit of n must be 1 or 9. So, $n = 10m \pm 1$ for some positive integer m, and $n^2 = (10m \pm 1)^2 = 100m^2 \pm 20m + 1$. So, the tenth digit of n^2 is an even number, and n^2 cannot be of the form $11, 111, 1111$, etc.

   It follows immediately that $4 \cdots 4 = 2^2(1 \cdots 1), 9 \cdots 9 = 3^2(1 \cdots 1)$.

   Finally, if $n^2 = 5 \cdots 5$, then $n = 10m + 5$ so that $n^2 = 100m^2 + 100m + 25$ so that the last two digits is 25, which is a contradiction.

   Hence, none of those numbers can be the square of an integer.