Math 307 Abstract Algebra **Homework 2** **Your name**

Solve the following problems (5 points each).

1. Prove that the set of all 2×2 upper triangular matrices with entries from $\mathbb{R}$ and determinant 1 is a group under matrix multiplication.

   (Extra 2 points if you can prove the results for $n \times n$ upper triangular matrices.)

2. Prove that the set $U(n)$ of elements in $\mathbb{Z}_n$ relatively prime to n form a group under multiplication $\mod n$.

   [Hint: If $a \in \mathbb{Z}_n$ satisfies $\gcd(a, n) = 1$, there is $x, y \in \mathbb{Z}$ such that $ax + ny = 1$.]

3. Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$.

4. Prove that in any group, an element and its inverse have the same order.

   Recall that the order of an element g in a group G is $n \in \{1, 2, 3, \ldots\}$ if n is the smallest positive integer such that $g^n = e$, the identity. If no such positive number, we say that g has infinite order.

   Hint: If g has order n, show that $(g^{-1})^n = e$ and no smaller positive number m will satisfy $(g^{-1})^m = e$.

5. Suppose that H is a proper subgroup of $\mathbb{Z}$ under addition and H contains $18, 30$ and 40, Determine H.

   [Hint: Find the smallest positive number in H.]

6. Suppose H_α is a subgroup of a group G for every $\alpha \in J$. Show that $\cap_{\alpha \in J} H_\alpha$ is a subgroup of G.

7. Let H and K be subgroups of a group G. Show that $H \cup K \leq G$ if and only if $H \leq K$ or $K \leq H$.

   [Hint: The ($\Leftarrow$) is clear. To prove ($\Rightarrow$), suppose $H \cup K \leq G$. Assume by contradiction that there is $h \in H - K$ and $k \in K - H$. Then $hk \in H \cup K$ and ....]

8. (Extra credit) Give an example of G with distinct proper subgroups such that H_1, H_2, H_3, whose union is a subgroup.