Math 307 Abstract Algebra **Homework 3** **your name**

Five points for each problem unless specified otherwise.

1. Let $H = \{a + bi : a, b \in \mathbb{R}, ab \geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ under addition.

2. Suppose H is a non-trivial subgroup of $(\mathbb{Z}, +)$. Show that H contains a positive integer, and therefore a smallest positive integer k. Then deduce that $H = k\mathbb{Z} = \{kr : r \in \mathbb{Z}\}$.

   Remark This shows that every non-trivial subgroup in $\mathbb{Z}$ is generated an element $k \in \mathbb{Z}^+$.

3. Suppose H is a non-trivial subgroup of $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ under addition, and $\overline{k}$ is the smallest positive integer in H when the elements of H are expressed in the form $\overline{r}$ with $0 \leq r < n-1$. Show the $H = \langle k \rangle$.

   Remark This shows that every non-trivial subgroup in $\mathbb{Z}_n$ is generated an element $\overline{k} \in \mathbb{Z}_n$.

4. Determine all the subgroups of $\mathbb{Z}_8$, and the subgroup lattice of $\mathbb{Z}_8$.

5. Suppose that a group contains elements a and b such that $|a| = 4, |b| = 2$, and $a^3b = ba$. Show that $|ab| = 2$.

6. (5 points for each part) Suppose G is a group with n elements, and H is a subgroup of G with m elements.

   (a) Suppose $H \neq G$ and $g_1 \in G - H$. Let $g_1H = \{g_1h : h \in H\}$.

   Show that $H \cap g_1H = \emptyset$ so that $|H \cup g_1H| = 2m$. (Here you need to argue $|g_1H| = m$.)

   (b) Suppose $H \cup g_1H \neq G$ and $g_2 \notin (H \cup g_1H)$.

   Show that $(H \cup g_1H) \cap g_2H = \emptyset$ so that $|H \cup g_1H \cup g_2H| = 3m$.

   (c) Show that G is a disjoint union of $H \cup g_1H \cup g_2H \cdots g_kH$ for some $g_1, \ldots, g_k \in G$

   so that n/m is a positive integer.

   Remark The set g_iH is called a left coset of the subgroup H. The fact that n/m is an integer is known as the Lagrange Theorem.

7. (5 points for each part) Use the result in the a last problem to prove the following.

   (a) A group of prime order must be cyclic.

   Hint: Let $a \in G$ not equal to the identity. Show that $\langle a \rangle = G$.

   (b) Let G be a group, and $a, b \in G$. If $|a|$ and $|b|$ are relatively prime, show that

   $$H = \langle a \rangle \cap \langle b \rangle = \{e\}.$$

   Hint: If $|H| = m > 1$ then ...

8. Suppose that $|G| = 24$ and that G is cyclic. If $a^8 \neq e$ and $a^{12} \neq e$, show that $G = \langle a \rangle$.

   Hint: Suppose $H = \langle a \rangle$. By the Lagrange Theorem, $|H| = ...$

9. (Extra credits) Suppose G is a set equipped with an associative binary operation $*$. Furthermore, assume that G has an left identity e, i.e., $e * g = g$ for all $g \in G$, and that every $g \in G$ has an left inverse g', i.e., $g' * g = e$. Show that G is a group.

   [Hint: Let $\hat{g}$ be the left inverse of g', where g' is the left inverse of $g \in G$. Show that $\hat{g} = g$ and conclude that $gg' = e$, i.e., the left inverse is also the right inverse. Then show that the left identity is also the right identity.]

10. (Extra credits) Let A be a set, and $\mathcal{P}(A)$ be its power set. Show that there is a group G with $|G| = |\mathcal{P}(A)|$,

    Hint: Case 1. $|A|$ is finite. Case 2. A is infinite. Let S_A be the group of bijections (permutations) on A under function composition. Then $|S_A| = |\mathcal{P}(A)|$.

    The Axiom of Choice is needed to show the $|\mathcal{P}(A)| = |S_A|$.