Math 307 Abstract Algebra   Homework 4

Solve the following problems. Five points for each problem.

1. Consider $\sigma = (13256)(23)(46512) \in S_5$. (Read the first 2 pages of Chapter 5 for the cycle notation.)

   (a) Express σ as a product of disjoint cycles.

   (b) Express σ as a product of transpositions.

   (c) (Extra 2 points.) Express σ as a product minimum number of transpositions.

   (Prove that the number is minimum!)

2. (a) Let $\alpha = (1, 3, 5, 7, 9, 8, 6)(2, 4, 10)$. Determine with proof the smallest positive integer n such that $\alpha^n = \alpha^{-5}$.

   (b) Let $\beta = (1, 3, 5, 7, 9)(2, 4, 6)(8, 10)$. If β^m is a 5-cycle, what can you say about m?

3. In S_7 show that $x^2 = (1, 2, 3, 4)$ has no solutions, but $x^3 = (1, 2, 3, 4)$ has at multiple solutions. (Extra 2 points if you determine all the solutions with proofs.)

   Hint: If $x = \sigma_1 \cdot \sigma_k$ is a solution in disjoint cycle representation, then ...

4. Describe in terms of the disjoint cycle decomposition all elements of order 5 in A_6.

5. Let $H \leq S_n$.

   (a) Show that either $H \leq A_n$ or $|H \cap A_n| = |H|/2$.

   (b) If $|H|$ is odd, show that $H \leq A_n$.

   Hint: (a) If $H \cap A_n \neq H$, set up a bijection from $H \cap A_n$ to $H - A_n$ in the latter case.

6. Let G be a group. Show that $\phi : G \to G$ defined by $\phi(g) = g^{-1}$ is an isomorphism if and only if G is Abelian.

7. Recall that $U(n)$ is group containing the elements in Z_n that are relatively prime to n such that $a * b = ab \pmod{n}$.

   Show that $\phi : U(16) \to U(16)$ defined by $\phi(x) = x^3$ is an isomorphism (automorphism). What about the maps $x \mapsto x^5$ and $x \mapsto x^7$?

8. Show that every isomorphism (automorphism) $\phi : (\mathbb{Q}, +) \to (\mathbb{Q}, +)$ has the form $\phi(x) = qx$ for $q = \phi(1)$.