Math 307 Abstract Algebra   Homework 5   Your name

Five points for each question.

1. Let $(\mathbb{R}^+, \cdot)$ be the group of positive real number under multiplication. Show that $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(x) = \sqrt{x}$ is an group isomorphism.

2. Show that the following pair of groups are not isomorphic.

   (a) $(\mathbb{Q}, +), (\mathbb{R}, +)$. (b) $(\mathbb{R}^+, \cdot), (\mathbb{R}^*, \cdot)$. (c) $(\mathbb{R}^*, \cdot), (\mathbb{C}^*, \cdot)$.

   [Recall that $\mathbb{R}^*$ and $\mathbb{C}^*$ are the sets of nonzero real numbers and nonzero complex numbers.]

3. Show that $G = \{e^{it} : t \in [0, 2\pi)\}$ under multiplication contains subgroups isomorphic to $(\mathbb{Z}, +)$ and $(\mathbb{Z}_n, +)$ for any $n \in \mathbb{N}$, and show that G is not a cyclic group.

4. Suppose ϕ_1, ϕ_2 are automorphisms of a group G. Show that $H = \{g \in G : \phi_1(g) = \phi_2(g)\}$ is a subgroup of G.

5. Let G be a group and $a \in G$. Suppose $|a| = n$ and the inner automorphism $\phi_a : G \to G$ defined by $\phi_a(x) = axa^{-1}$ has order m in $\mathrm{Aut}(G)$. Show that $m|n$.

   Give an example of G and a so that $1 < m < n$.

   [Hint: Show that $(\phi_a)^n$ is the identity map, and ....]

6. Suppose G is a group of order n, and $k \in \mathbb{N}$ is relatively prime to n. Show that $g : G \to G$ defined by $g(x) = x^k$ is one-one. If G is Abelian, show that g is an automorphism.

   [Hint: To prove that g is bijective, it suffices to show that g is 1-1. It suffices to show that $g(x) = e$ if and only if $x = e$.]

7. Suppose $1 \le i < j \le n$.

   (a) Show that $(i, j) = (j, j-1)(j-1, j-2) \cdots (i+1, i)(i+1, i+2) \cdots (j-1, j)$.

   (b) Show that every element $\sigma \in S_n$ is a product of transpositions of the form

   $$(1, 2), (2, 3), \ldots, (n-1, n).$$

   [Hint: To prove (a), show that the bijections on right sides will exchange i and j, and fixes all other k.]

8. (Extra 5 points) (a) Show that every $\sigma \in S_n$ is a product of the n-cycle $\alpha = (1, 2, \ldots, n)$ and $\tau = (1, 2)$.

   [Hint: It suffices to use α, τ to generate all transpositions of the form $(i, i+1)$ for $i = 1, \ldots, n-1$. To this end, show that for $k = 1, \ldots, n-1$, $\alpha^k \tau \alpha^{-k} = (k, k+1)$, equivalently, $\alpha^k \tau = (k, k+1)\alpha^k$]

   (b) (Open problem) In (a), determine the minimum number of α and τ needed for a given σ.

   (c) (Conjecture) For $n > 3$, every σ is a product of no more than $\binom{n}{2}$ permutations in the set $\{\alpha, \alpha^{-1}, \tau\}$.