Math 307 Abstract Algebra Homework 6 Sample solution

1. (a) Let $H = \langle(1,2,3)\rangle \in A_4$. Write down all the left cosets of H in A_4, and the right cosets of H in S_4.

   (b) Let $H = \{e^{it} : t \in [0, 2\pi)\} \leq \mathbb{C}^*$. Describe geometrically the left cosets of H.

   Solution. (a) The three other left cosets of H in A_4 are: $(124)H = \{(124), (14)(32), (134)\}$,

   $$(142)H = \{(142), (234), (13)(24)\},\ (143)H = \{(143), (12)(34), (243)\}.$$

   The seven other right cosets of H in S_4 are: $H(12) = \{(12), (13), (23)\}$,

   $$H(14) = \{(14), (1423), (1432)\},\ H(24) = \{(24), (1243), (1324)\},$$

   $$H(34) = \{(34), (1234), (1432)\},\ H(124) = \{(124), (13)(24), (243)\},$$

   $$H(134) = \{(134), (234), (12)(34)\},\ H(143) = \{(143), (14)(23), (142)\}.$$

   (b) Note that for every $r > 0$, $rH = \{re^{it} : t \in [0, 2\pi)\}$ that corresponds a circle with radius r in the complex plane. These are all the cosets of H as they form a disjoint union of $\mathbb{C}^*$.

2. Suppose K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

   Solution. If $|H| = m$, then $42 < m < 420$ such that m is a multiple of 42 and a factor of 420. So, $m = 42 * 2 = 84$ or $42 * 5 = 210$.

3. Let G be a group with $|G| = pq$, where p, q are primes. Prove that every proper subgroup of G is cyclic. Give an example to show that such a group G may not be cyclic.

   Solution. Let $H \leq G$ and $H \neq G$. Then $|H| = \{1, p, q\}$. So, $H = \{e\}$ or $|H|$ has prime order. In either case, H is cyclic. Let $G = S_3$ has $6 = 2 * 3$ elements. Then every proper subgroup of G is cyclic. But there is no element in G has order 6. So, G is not cyclic.

4. Let G be a group of order p^2 for a prime p. Show that G is cyclic or $g^p = e$ for all $g \in G$.

   Solution. Let $g \in G$. Then $|g| \in \{1, p, p^2\}$. If there is $g \in G$ with order p^2, then $G = \langle g\rangle$ is cyclic. Otherwise, every element $g \in G$ have order in $\{1, p\}$ so that $g^p = e$.

5. Show that a group G of order 55 cannot have exactly 20 elements of order 11. Give a reason for your answer.

   Solution. We can show that if G cannot have more than 10 elements of order 11. If it does, then we can let a be such an element so that $H = \langle a\rangle = \{a, a^2, \dots, a^{10}, e\}$. Note that all $a, \dots, a^{10}$ have order 11. Because there are more than 10 elements of order 11, there is $b \notin H$ with order 11. Let $K = \{b, b^2, \dots, b^{10}, e\}$. Note that every element in H other than e is a generator of H. Similarly, every element in K other than e is a generator of K. So, $H \cap K = \{e\}$. Otherwise, we have $x \in H \cap K$ with $x \neq e$ such that $H = K = \langle x\rangle$, which is impossible. But then $HK = \{hk : h \in H, k \in K\}$ has $|H| \cdot |K|/|H \cap K| = 121$ elements in G, which is a contradiction.

6. Let G be a group, and $H \le K \le G$. Suppose $a_1K, \dots, a_rK$ are distinct cosets of K in G, and $b_1H, \dots, b_sH$ are distinct cosets of H in K. Prove that a_ib_jH with $1 \le i \le r, 1 \le j \le s$ are all the distinct cosets of H in G. Deduce that

$$|G : H| = |G : K|\,|K : H|.$$

Solution. Note that we have the two disjoint unions relations:

$$G = a_1K \cup \cdots \cup a_rK, \quad K = b_1H \cup \cdot \cup b_sH.$$

We will prove that (a) $G = \cup_{1\le i\le r, 1\le j\le s} a_rb_sH$, and (b) $a_ib_jH \cap a_pb_qH = \emptyset$ if $(i, j) \ne (p, q)$. Then we can conclude that G has $rs = |G : K| \cdot |K : H|$ for many distinct left cosets of H.

For (a) note that $g \in G$ implies $g \in a_iK$ for some $1 \le i \le r$ so that $g = a_ik$ for some $k \in K$. Since $a_i^{-1}g \in K$, we see that $a_i^{-1}g \in b_jH$ for some for some $1 \le j \le s$. Thus, $g = a_ib_jh \in a_ib_jH$. So, $G \subseteq \cup_{1\le i\le r, 1\le j\le s} a_rb_sH$. The reverse inclusion is clear.

For (b), suppose $(i, j) \ne (p, q)$. Consider two cases.

(b.1) $i \ne p$. Then $a_ib_jH \subseteq a_iK$ and $a_pb_qH \subseteq a_pK$. So, $a_ib_jH \cap a_pb_qH \subseteq a_iK \cap a_pK = \emptyset$.

(b.2) $i = p$ and $j \ne q$. Suppose $x \in a_ib_jH \cap a_ib_qH$. Then $x = a_ib_jh_1$ and $x = a_ib_qh_2$ for some $h_1, h_2 \in H$. By cancellation, we have $y = b_jh_1 = b_qh_2 \in b_jH \cap b_qH$, which is a contradiction because $b_jH \cap b_qH = \emptyset$.

7. (a) Prove that if $H \le G$ and $|G : H| = 2$, then H is normal.

(b) Deduce that if $H \le S_n$ contains a an odd permutation, then H has a normal subgroup.

Solution. (a) Suppose $|G : H| = 2$. If $a \in H$, then $aH = H = Ha$. If $a \notin H$, then $aH = G - H = Ha$. Thus, H is normal.

(b) If $H \le S_n$, then by a previous homework, we know that $H \le A_n$ or $|H \cap A_n| = |H|/2$. Because H contains an odd permutation, we see that the latter case holds. Then $H \cap A_n$ will be a subgroup of H of index 2, and is normal.

8. Let $H \le G$.

(a) Prove that the map $f : aH \to Ha$ defined by $f(ah) = ha$ is a bijection.

(b) Prove that H is normal if and only if $aHa^{-1} \subseteq H$ for all $a \in G$.

Solution. (a) For any $ah \in aH$, we have $f(ah) = ha \in Ha$. So, the map is well defined. Suppose $ah_1, ah_2 \in aH$ satisfy $f(ah_1) = f(ah_2)$, i.e., $h_1a = h_2a$. By cancellation, $h_1 = h_2$ so that $ah_1 = ah_2$. So, f is one-one. Suppose $ha \in Ha$. Then $ah \in aH$ and $f(ah) = ha$. So, f is onto.

(b) If H is normal, then $Ha = aH$ for all $a \in G$. Suppose $x = aha^{-1} \in aHa$, then $ah \in aH = Ha = \{xa : x \in H\}$ and thus, $ah = \hat{h}a$ for some $\hat{h} \in H$. So, $x = aha^{-1} = \hat{h}aa^{-1} = \hat{h} \in H$.

Conversely, suppose $aHa^{-1} \subseteq H$ for all $a \in G$. Let $g \in G$. Because $gHg^{-1} \subseteq H$, $gH = (gHg^{-1})g = \{(ghg^{-1})g : h \in H\} \subseteq \{\hat{h}g : \hat{h} \in H\} = Hg$ so that we have $gH \subseteq Hg$. Because $g^{-1}Hg \subseteq H$, $Hg = g(g^{-1}Hg) \subseteq gH$. So, $gH = Hg$ for any $g \in G$, i.e., H is normal.

9. (Extra credits) Prove that A_5 has no subgroup of order 30.

Solution Note that A_5 has elements of the form in disjoint cycle decomposition:

(1) ε, (2) $(i_1, i_2)(j_1, j_2)$ (15 of them), (3) (i_1, i_2, i_3) (20 of them), (4) $(i_1, \dots, i_5)$ (24 of them).

Suppose $H \leq A_5$ has order 30 so that it is normal in A_5 by Problem 7. Note that $\sigma = (i_1 i_2 i_3) \in H$ implies $\sigma^2 = (i_1 i_3 i_2) \in H$; $\sigma = (i_1, \dots, i_5) \in H$ implies that $\sigma, \sigma^2, \dots, \sigma^4 \in H$. Thus $30 = 1 + n_2 + 2n_3 + 4n_4$, where there are one element in H of order 1, n_2 elements of order 2, $2n_3$ elements of order 3, $4n_4$ elements of order 4. Now, $30 = 1 + n_2 + 2n_3 + 4n_4$ is even. So, $n_2 > 0$. Let $\sigma = (i_1, i_2)(j_1, j_2) \in H$.

Consider $\tau = (i_1, i_2, j_1) \in A_4$. Then $\tau\sigma\tau^{-1} = (j_1, i_2)(i_1, j_2) \in H$ by Problem 8. Similarly, $\tau^{-1}\sigma\tau = (j_2, i_2)(j_1, i_1) \in H$. But then $K = \{\varepsilon, \sigma, \tau^{-1}\sigma\tau, \tau\sigma\tau^{-1}\}$ is a 4 element subgroup of H, which is impossible by Lagrange Theorem.