Math 307 Abstract Algebra Homework 6 Your name

Five points for each question.

1. (a) Let $H = \langle (1,2,3) \rangle \in A_4$. Write down all the left cosets of H in A_4, and the right cosets of H in S_4.

   (b) Let $H = \{e^{it} : t \in [0, 2\pi)\} \leq \mathbb{C}^*$. Describe geometrically the left cosets of H.

2. Suppose K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

3. Let G be a group with $|G| = pq$, where p, q are primes. Prove that every proper subgroup of G is cyclic. Give an example to show that such a group G may not be cyclic.

4. Let G be a group of order p^2 for a prime p. Show that G is cyclic or $g^p = e$ for all $g \in G$.

5. Show that a group of order 55 cannot have exactly 20 elements of order 11? Give a reason for your answer.

   [Hint: If G is cyclic, then number of elements of order 11 equal???

   If G is not cyclic, then $a \in G$ has order 1, 5, or 11. So, ...]

6. Let G be a group, and $H \leq K \leq G$. Suppose $a_1K, \ldots, a_rK$ are distinct cosets of K in G, and $b_1H, \ldots, b_sH$ are distinct cosets of H in K. Prove that a_ib_jH with $1 \leq i \leq r, 1 \leq j \leq s$ are all the distinct cosets of H in G. Deduce that

$$|G : H| = |G : K|\,|K : H|.$$

   Recall that $H \leq G$ is a normal subgroup if $aH = Ha$ for all $a \in G$.

7. (a) Prove that if $H \leq G$ and $|G : H| = 2$, then H is normal.

   (b) Deduce that if $H \leq S_n$ contains a an odd permutation, then H has a normal subgroup.

8. Let $H \leq G$.

   (a) Prove that the map $f : aH \to Ha$ defined by $f(ah) = ha$ is a bijection.

   (b) Prove that H is normal if and only if $aHa^{-1} \subseteq H$ for all $a \in G$.

9. (Extra credits) Prove that A_5 has no subgroup of order 30.

   [Hint: Prove by contradiction. Assume $H \leq A_5$ has 30 elements. Then $A_n - H$ is a left coset as well as a right coset of H in A_n. Argue that H has an element $\sigma_1 = (i_1, i_2)(j_1, j_2)$, and then show that $\sigma_2 = (i_1, j_2), (i_2, j_1), \sigma_3 = (i_1, j_1)(i_2, j_2) \in H$. Then argue that $\{\varepsilon, \sigma_1, \sigma_2, \sigma_3\}$ is a 4-element subgroup of H, ...]