Math 307 Abstract Algebra   Homework 7   Name: Sample Texfile

Five points for each question.

1. If r is a divisor of m and s is a divisor of n, find a subgroup of $\mathbb{Z}_m \oplus \mathbb{Z}_n$ that is isomorphic to $\mathbb{Z}_r \oplus \mathbb{Z}_s$.

   [Example: In $\mathbb{Z}_6 \oplus \mathbb{Z}_6$, the subgroup $\{(2p, 3q) : p, q \in \mathbb{Z}\}$ is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_2$ under the isomorphism $\phi(2p, 3q) = (p, q) \in \mathbb{Z}_3 \oplus \mathbb{Z}_2$.]

2. (a) What is the order of the element $14 + \langle 8 \rangle$ in $\mathbb{Z}_{24}/\langle 8 \rangle$?

   (b) What is the order of $4U_5(105)$ in the factor group $U(105)/U_5(105)$.

   [Hint: The order of aH is the smallest $m \in \mathbb{N}$ such that $a^m \in H$.]

3. Let $G = \mathbb{Z}_4 \oplus U(4)$, $H = \langle (2, 3) \rangle$ and $K = \langle (2, 1) \rangle$. Show that $G/H \ncong G/K$.

   [Hint: $H = \{(2, 3), (0, 1)\}$ and $K = \{(2, 1), (0, 1)\}$. Show that G/H is cyclic but G/K is not.]

4. Let G be a finite group, and H be a normal subgroup of G.

   (a) Show that the order of aH in G/H must divide the order of a in G.

   (b) Show that it is possible that $aH = bH$, but $|a| \neq |b|$.

   [Hint: (a) Note that if $a^n = e$, then $(aH)^n = eH = H$.]

   (b) Note that if $a \in H$ then $aH = eH$.]

5. If G is a group and $|G : Z(G)| = 4$, prove that $G/Z(G)$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

   [Hint: Note that a four element group is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.]

6. Suppose that $N \lhd G$ and $|G/N| = m$, show that $x^m \in N$ for all $x \in G$.

7. (a) Let $G = \{3^a 6^b 10^c : a, b, c \in \mathbb{Z}\}$ under multiplication. Show that G is isomorphic to $\langle 3 \rangle \times \langle 6 \rangle \times \langle 10 \rangle$.

   (b) Let $H = \{3^a 6^b 12^c : a, b, c \in \mathbb{Z}\}$ under multiplication. Show that H is NOT isomorphic to $\langle 3 \rangle \times \langle 6 \rangle \times \langle 12 \rangle$.

   [Hint: Check internal direct product conditions.]

8. (Extra credits) (a) (5 points) Show that $U(2^3) = \langle 7 \rangle \oplus \langle 3 \rangle$, $U(2^4) = \langle 15 \rangle \oplus \langle 3 \rangle$.

   [Note: $\langle 7 \rangle, \langle 3 \rangle$ are subgroups in $U(2^3)$ generated by 7 and 3 in the first part, and $\langle 15 \rangle, \langle 3 \rangle$ are subgroups in $U(2^4)$ generated by 7 and 3 in the second part.]

   (b) (5 points) Show that $U(2^m) = \langle 2^m - 1 \rangle \oplus \langle 3 \rangle$.

9. (Extra credits) Let $n = p_1^{r_1} \cdots p_k^{r_k}$ such that $p_1, \ldots, p_k$ are distinct primes and $r_1, \ldots, r_k$ are positive integer. Determine n such that $U(n)$ is a cyclic group. [Give explanation.]

10. (Extra credits) Suppose s, t are relatively prime. Prove that $U_t(st)$ is isomorphic to $U(s)$.

    [Hint: $U_t(st) = \{kt + 1 \in U(st) : k \in \{0, \ldots, s - 1\}\}$. Note that some of the k should would not give $kt + 1 \in U(st)$.

Define $f : U_t(st) \to U(s)$ by $f(x) = [x]_s = y$, i.e., $x = sq + y$ with $y \in \{0, \ldots, s-1\}$. We need to show that f is an isomorphism. One important step is to show that $\gcd(x, st) = 1$ will ensure that $\gcd(y, s) = 1$ so that $y \in U(s)$.

To show that f is surjective, for any $y \in U(s)$, we need to find $x = pt + 1 \in U_t(st)$ such that $f(x) = y$. Thus, we need to find $[y]_s = [pt + 1]_s = [pt]_s + [1]_s$. Note that $[y]_s \in U(s)$ has an inverse in $U(s)$. Also, $[t]_s \in U(s)$ also has an inverse in $U(s)$. With these two pieces of information, one should be able to find k such that $x = ks + 1 \in U(st)$ satisfying $[x]_s = [y]_s \in U(s)$.]