Math 307 Abstract Algebra Homework 8 Sample Solution

1. Let G be the group of nonzero real numbers under multiplication. Suppose r is a positive integer. Show that $x \mapsto x^r$ is a homomoprhism. Determine the kernel, and determine r so that the map is an isomorphism.

   Solution. Note that $\phi(xy) = (xy)^r = x^r y^r = \phi(x)\phi(y)$ for all $x, y \in G$. Thus, ϕ is an homomorphism.

   Note that $ker(\phi) = \{x \in G : \phi(x) = 1\} = \{1\}$ if r is odd, and $ker(\phi) = \{1, -1\}$ if r is odd.

   We claim that ϕ is isomorphism if r is an odd positive integer. Since $\text{Ker}(\phi)$ is a singleton, it remains to show that ϕ is surjective. In fact, in such a case, for any $y \in G$, we can let $x = y^{1/r}$ so that $\phi(x) = y$. The result follows.

2. Let G be the group of polynomial in x with real coefficients under addition. Define the map $p(x) \mapsto P(x) = \int p(x)$ such that $P(0) = 0$. Show that f is an homomorphism, and determine its kernel.

   Solution. By the theory of integration, we see that for any two polynomials $f(x), g(x)$, we have

   $$\phi(f(x) + g(x)) = \int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx = \phi(f(x)) + \phi(g(x))$$

   if we set the integration constant to be zero. So, ϕ is a homomorphism.

   Note that $f(x) = \sum_{j=0}^{n} a_j x^n$, we have $\phi(f(x)) = \sum_{j=0}^{n} \frac{a_j}{j+1} x^{j+1} = 0$ if and only if $a_0, \ldots, a_n = 0$, i.e., $f(x) = 0$. Thus, $ker(\phi) = \{0\}$.

3. How many homomorphisms are there from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$? How many of them are surjective? Explain your answer.

   Solution. We prove a general result. Note that a homomorphism $\phi : \mathbb{Z}_m \to \mathbb{Z}_n$ is completely determined by $\phi([1]_m) = [k]_n$ with $k = 0, 1, \ldots, n-1$. In order that ϕ is well-defined, $[x]_m = [y]_m$ should ensure $[kx]_n = [ky]_n$. The condition reduces to: $m|(x - y)$ implies $n|k(x - y)$, equivalently, $n|km$. It will be surjective if $\phi([i]) = [1]$ for some i because we can then get $\phi([xi]) = [x]$ for every $x \in \mathbb{Z}_n$.

   Thus, $\phi : \mathbb{Z}_{20} \to \mathbb{Z}_8$ is a homomorphism is a homomorphism with $\phi([1]) = [k]$ if and only if $k = 0, 2, 4, 6$. None of these are surjective because the range space equals $\phi(\mathbb{Z}_{20}) = \langle \phi(1) \rangle = \langle k \rangle$, which cannot be $\mathbb{Z}_8$ for $k = 0, 2, 4, 6$.

4. Prove that $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $\phi(a, b) = a - b$ is a homomorphism. Determine the kernel, and $\phi^{-1}(\{3\}) = \{(x, y) \in \mathbb{Z} \oplus \mathbb{Z} : \phi(x, y) = 3\}$.

   Solution. Note that $\phi(x_1, x_2) + \phi(y_1, y_2) = (x_1 - x_2 + (y_1 - y_2) = (x_1 + y_2) - (x_2 + y_2) = \phi(x_1 + y_1, x_2 + y_2)$ for all $(x_1, x_2), (y_1, y_2) \in \mathbb{Z} \oplus \mathbb{Z}$. So, ϕ is a homomorphism.

   Now, $ker(\phi) = \{(x_1, x_2) : \phi(x_1, x_2) = x_1 - x_2 = 0\} = \{(x, x) : x \in \mathbb{Z}\}$.

   Note that $\phi(3, 0) = 3 - 0 = 3$. Thus, $\phi^{-2}(\{3\}) = (3, 0) + ker(\phi) = \{(3 + x, x) : x \in \mathbb{Z}\}$.

5. (a) Explain why $x \mapsto 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.

   (b) Prove that there is no isomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

   Solution. (a) Suppose $\phi(x) = 3x$. If $[x]_{12} = [y]_{12}$, i.e., $x - y = 12q$ for some $q \in \mathbb{Z}$, then $3x - 3y = 3(12q) = 36q$. If we let $(x, y) = (12, 0)$, then $3x - 3y = 36q$ is not divisible by 10 so that $[3x]_{10} \neq [3y]_{10}$. In fact, in this case $[3x]_{10} = [36]_{10} = [6]_{10}$ and $[3y]_{10} = [0]_{10}$.

   (b) If ϕ is an isomorphism, then $\phi(1, 0) = (a, b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ has the same order as $(1, 0) \in \mathbb{Z}_8 \oplus \mathbb{Z}_2$. Now, $|(1, 0)| = \text{lcm}(8, 0) = 8$, and $|(a, b)| = \text{lcm}(|a|, |b|)$ is a factor of $\text{lcm}(4, 4) = 4$. Since there is no $(a, b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_4$ with order 8, we see that there is not isomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

6. For each pair of positive integer m and n, show that the map from $\mathbb{Z}$ to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ defined by $x \mapsto ([x]_m, [x]_n)$ is a homomorphism.

   (a) Determine the kernel when $(m, n) = (3, 4)$.

   (b) Determine the kernel when $(m, n) = (6, 4)$.

   (c) Generalize the result to general (m, n).

   Solution. Note that $\phi(x+y) = ([x+y]_m, [x+y]_n) = ([x]_m + [y]_m, [x]_n + [y]_n) = ([x]_m, [x]_n) + ([y]_m, [y]_n) = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{Z}$. Thus, ϕ is an homomorphism.

   (a) Note that $ker(\phi) = \{x \in \mathbb{Z} : ([x]_3, [x]_4) = ([0]_3, [0]_4)\} = \{12q : q \in \mathbb{Z}\}$.

   (b) Note that $ker(\phi) = \{x \in \mathbb{Z} : ([x]_6, [x]_4) = ([0]_3, [0]_4)\} = \{12q : q \in \mathbb{Z}\}$.

   (c) In general, if $k = \text{lcm}(m, n) = k$, then $ker(\phi) = \{kq : q \in \mathbb{Z}\}$.

7. Show that if $\phi : G_1 \to G_2$ is an homomorphism, and K is a normal subgroup of G_2, then $H = \phi^{-1}(K) = \{x \in G_1 : \phi(x) \in K\}$ satisfies $aHa^{-1} = H$ for all $a \in G_1$.

   Solution. First, note that $\phi(e_1) = e_2 \in K$. So, $e_1 \in \phi^{-1}(K)$. If $a, b \in \phi^{-1}(K)$, we have $\phi(a), \phi(b) \in K$. Because K is a subgroup, $\phi(a)\phi(b)^{-2} \in K$. Thus, $\phi(ab^{-1}) = \phi(a)\phi(b)^{-2} \in K$ so that $ab^{-1} \in \phi^{-1}(K)$. Hence, $\phi^{-1}(K)$ is a subgroup.

   Now, for any $a \in \phi^{-1}(K)$ and $g \in G_1$, we have $\phi(gag^{-1}) = \phi(g)\phi(a)\phi(g)^{-1} \in K$ because K is normal in G_2. Thus, $gag^{-1} \in \phi^{-1}(K)$.

   Combining, we see that $\phi^{-1}(K)$ is normal in G_1.

8. Determine all homomorphisms and **isomorphisms** from $\mathbb{Z}_n$ to itself.

   Solution. Note that ϕ is determined by $\phi(1) = k$, and must have a formula $\phi(x) = kx$.

   Now, we can choose $k = 1, \dots, n$, and for each choice gives rise to a different ϕ as $\phi(1)$ are all different. Moreover, $\phi(x+y) = k(x+y) = kx + ky = \phi(x) + \phi(y)$ for all $x, y \in \mathbb{Z}_n$. Moreover, if $[x]_n = [y]_n$, i.e., $x - y = nq$, then $k(x - y) = knq$ so that $[\phi(x)]_n = [kx]_n = [ky]_n = [\phi(y)]_n$. So, $\phi(x) = kx$ is a homomorphism for $k = 1, \dots, n$.

   In order for ϕ to be an isomorphism, we requires $\phi(\mathbb{Z}_n) = \langle \phi(1) \rangle = \langle \phi(1) \rangle$. This is true if and only if $\gcd(k, n) = 1$.

9. Suppose $K \leq G$ and $N \triangleleft G$. Show that KN is a subgroup of G and N is a normal subgroup of KN such that KN/N is isomorphic to $K/(K \cap N)$.

   Solution. Note that $NK = \{kn : k \in K, n \in N\}$. Note that $e_1 = e_1e_1 \in NK$. If $x = k_1n_1, y = k_2n_2 \in KN$ with $k_1, k_2 \in K$ and $n_1, n_2 \in N$, then $xy^{-1}k_1n_1(k_2n_2)^{-1} = k_1n_1n_2^{-1}k_2^{-1} = k_1k_2^{-1}(k_2n_1n_2^{-1}k_2^{-1}) \in KN$ because $k_1k_2^{-2} \in K$, $n_1n_2^{-1} \in N$ and thus $k_2(n_1n_2^{-1})k_2^{-1} \in N$.

   Now, if $x \in N$ and $g \in KN$, then $gxg^{-1} \in N$ as N is normal in G. So, N is normal in KN.

   It is easy to show that $K \cap N$ is a subgroup of N. If $x \in K \cap N$ and $g \in K$, then $x \in K$ and $x \in N$ so that $gxg^{-1} \in K$ and $gxg^{-1} \in N$ as N is normal. Thus, $x \in K \cap N$. Hence $K \cap N$ is normal in K.

   Now, consider $\Phi : KN/N \to K/(K \cap N)$ so that $\Phi(knN) = \Phi(kN) = k(K \cap N)$. If $aN = bN$ in KN/N with $a, b \in K$, then $a^{-1}b \in K$ and $a^{-1}b \in N$ so that $a^{-1}b \in K \cap N$. Thus, $a(K \cap N) = b(K \cap N)$. Hence, Φ is well-defined.

   Now, $\Phi(aNbN) = \Phi(abN) = ab(K \cap N) = [a(K \cap N)][b(K \cap N)] = \Phi(aN)\Phi(bN)$. So, Φ is a homomorphism.

   To show that Φ is one-one, note that $\Phi(aN) = a(K \cap N) = K \cap N$ implies that $a \in K \cap N$ so that $aN = N$. Thus, $ker(\Phi) = \{N\}$.

   To show that Φ is onto, note that $a(L \cap N)$ is the image of $\Phi(aN)$. So, Φ is onto.

10. (Optional) Suppose M and N are normal subgroup of G and $N \leq M$, Show that the $\phi : G/N \to G/M$ defined by $\phi(gN) = gM$ is a well defined surjective group homorphism with $Ker(\phi) = M/N$.

    Solution. Define $\phi : G/N \to G/M$ by $\phi(gN) = gM$. If $aN = bN$, then $a^{-1}b \in N \leq M$ so that $aM = bM$.

    Now, $\phi(aNbN) = \phi(abN) = abM = aMbM = \phi(aN)\phi(bN)$ for any $aN, bN \in G/N$.

    For any $aM \in G/M$, we have $\phi(aN) = aM$. So, ϕ is onto.

    Now, $ker(\phi) = \{aN : \phi(aN) = aM = M\} = \{aN : a \in M\} = M/N$.

    By the group isomorphism theorem, we see that $(G/N)/M/N)$ is isomorphic to G/M.