Math 307 Abstract Algebra   Homework 8   Sample Texfile

Five points for each question.

1. Let G be the group of nonzero real numbers under multiplication. Suppose r is a positive integer. Show that $x \mapsto x^r$ is a homomoprhism. Determine the kernel, and determine r so that the map is an isomorphism.

   Note: If ϕ is an isomorphism, then $Ker(\phi) = \{1\}$.

2. Let G be the group of polynomial in x with real coefficients under addition. Define the map $p(x) \mapsto P(x) = \int p(x)$ such that $P(0) = 0$. Show that f is an homomorphism, and determine its kernel.

   Example: If $f(x) = 1 + x - x^2$, then $\phi(f(x)) = x + x^2/2 - 3x^3/3$.

3. How many homomorphisms are there from $\mathbb{Z}_{20}$ to $\mathbb{Z}_8$? How many of them are surjective? Explain your answer.

   Note: The structure of $\phi : \mathbb{Z}_m \to \mathbb{Z}_n$ is completely determined by $\phi(1)$. Reason: If $\phi(1) = k$, then the homomorphism will be given by the formula $\phi(x) = \phi(1) + \cdots + \phi(1) = kx$.

4. Prove that $\phi : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ by $\phi(a, b) = a - b$ is a homomorphism. Determine the kernel, and $\phi^{-1}(\{3\}) = \{(x, y) \in \mathbb{Z} \oplus \mathbb{Z} : \phi(x, y) = 3\}$.

5. (a) Explain why $x \mapsto 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.

   (b) Prove that there is no isomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

   Hint: Consider the order of $\phi(1, 0)$ if an isomorphism ϕ exists.

6. For each pair of positive integer m and n, show that the map from $\mathbb{Z}$ to $\mathbb{Z}_m \oplus \mathbb{Z}_n$ defined by $x \mapsto ([x]_m, [x]_n)$ is a homomorphism.

   (a) Determine the kernel when $(m, n) = (3, 4)$.

   (b) Determine the kernel when $(m, n) = (6, 4)$.

   (c) Generalize the result to general (m, n).

7. Show that if $\phi : G_1 \to G_2$ is an homomorphism, and K is a normal subgroup of G_2, then $H = \phi^{-1}(K) = \{x \in G_1 : \phi(x) \in K\}$ satisfies $aHa^{-1} = H$ for all $a \in G_1$.

   [Remark: This shows that $\phi^{-1}(K)$ is normal in G_1. You may use the proof shown in class.]

8. Determine all homomorphisms and **isomorphisms** from $\mathbb{Z}_n$ to itself.

9. Suppose $K \leq G$ and $N \lhd G$. Show that KN is a subgroup of G and N is a normal subgroup of KN such that KN/N is isomorphic to $K/(K \cap N)$.

   [Hint: You can assume that N is normal in KN, and $K \cap N$ is normal in K, and construct $\phi : KN/N \to K/(K \cap N)$ by $\phi(knN) = \phi(kN) = k(K \cap N)$ for any $kn \in KN$.]

10. (Optional) Suppose M and N are normal subgroup of G and $N \leq M$, Show that the $\phi : G/N \to G/M$ defined by $\phi(gN) = gM$ is a well defined surjective group homorphism with $Ker(\phi) = M/N$.

    [Remark: Consequently, $M/N \lhd G/N$ and $(G/N)/(M/N) = G/M$.