Math 307 Abstract Algebra   Homework 9

Five points for each question unless specified otherwise.

1. Suppose G is an Abelian group. Show that $H = \{g \in G : |g| \text{ is finite}\}$ is a subgroup.

2. Suppose G is a finite Abelian group, and $m||G|$. Show that G has a subgroup of order m.

   [Hint: Up to isomorphism, we may assume $G = \mathbb{Z}_{p_1^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p_k^{r_k}}$, where $p_1, \dots, p_k$ are primes, so that $n = p_1^{r_1} \cdots p_k^{r_k}$. If $m|n$, we may assume that $m = p_1^{t_1} \cdots p_k^{t_k}$ with $0 \le t_j \le r_j$ for $j = 1, \dots, k$. Show that there is a subgroup of the form $H_1 \oplus \cdots \oplus H_k$ in $G = G_1 \oplus \cdots \oplus G_k$ with order m, where $G_1 = \langle(1, 0, \dots, 0)\rangle, G_2 = \langle(0, 1, 0, \dots, 0)\rangle$, etc.]

3. (10 points) Compare the number of isomorphic classes of subgroups of an Abelian group of orders m and n for each of the following if p, q are primes, and $r \in \mathbb{N}$.

   (a) $n = 3^2, m = 5^2$.

   (b) $n = 2^4, m = 5^4$,

   (c) $n = p^r, m = q^r$,

   (d) $n = p^r$ and $m = p^r q$,

   (e) $n = p^r$ and $m = p^r q^2$.

   [Hint: For each part, you only need to decide whether they have the same number of isomoprhism classes, or twice as many, etc.]

4. (a) Give an example of a subset of a ring that is a subgroup under addition but not a subring.

   (b) Give an example of a finite non-commutative ring.

5. Show that if m, n are integers and a, b are elements in a ring. Then $(ma)(nb) = (mn)(ab)$. [Note that for positive m, $ma = a + \cdots + a$ (m times); for $(-m)a = m(-a) = (-a) + \cdots + (-a)$.]

   [Hint: You might want to discuss the cases (1) $m, n > 0$, (2) $mn = 0$, (3) $mn < 0$, (4) $m, n < 0$.]

6. Let R be a ring.

   (a) Suppose $a \in R$. Show that $S = \{x \in R : ax = xa\}$ is a subring.

   (b) Show that the center of R defined by $Z(R) = \{x \in R : ax = xa \text{ for all } a \in R\}$ is a subring.

7. Let R be a ring.

   (a) Prove that R is commutative if and only if $a^2 - b^2 = (a + b)(a - b)$ for all $a, b \in R$.

   (b) Prove that R is commutative if $a^2 = a$ for all $a \in R$.

   (Such a ring is called a Boolean ring.)

   [Hint: In (b), think about $a = (x + y), (x - y)$, etc.

8. Give an example of a Boolean ring with 4 elements. Give an example of a Boolean ring with infinitely many elements.

   [Hint: Consider $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and extend the idea.]