Math 307 Abstract Algebra **Homework 10** **Your name**

Five points for each question.

1. Show that every nonzero element of $\mathbb{Z}_n$ is a unit (element with multiplicative inverse) or a zero-divisor.

   Hint: Consider $\mathbb{Z}_6$ to get some insight.

2. Show that every nonzero element in $\mathbb{Z}_7[i] = \{a + bi : a, b \in \mathbb{Z}_7\}$ has a multiplicative inverse.

   [Hint: Show that for any nonzero $a+ib$, we can find $(x+iy)$ such that $(a+ib)(x+iy) = 1$, i.e., $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so that $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (a^2+b^2)^{-1}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now, argue that $(a^2+b^2)^{-1}$ always exists if $a + ib \neq 0$.]

   (Extra 5 points.) Show that not every nonzero element in $\mathbb{Z}_p[i]$ has a multiplicative inverse for a given prime p.

   (Extra 5 points.) Determine those prime p such that every nonzero element in $\mathbb{Z}_p[i]$ has a multiplicative inverse.

3. (a) Given an example of a commutative ring without zero-divisors that is not an integral domain.

   (b) Find two elements a and b in a ring such that a, b are zero-divisors, $a + b$ is a unit.

4. (a) Give an example to show that the characteristic of a subring of a ring R may be different from that of R.

   (b) Show that the characteristic of a subdomain of an integral domain D is the same as that of D.

5. An element a of a ring R is nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$.

   (a) Show that if a and b are nilpotent elements of a commutative ring, then $a + b$ is also nilpotent.

   (b) Show that a ring R has no nonzero nilpotent element if and only if 0 is the only solution of $x^2 = 0$ in R.

   [Hint: (a) Assume $a^n = 0$ and $b^m = 0$. Show that $(a + b)^k = 0$ for some k.

   (b) Consider the special case: if $a^4 \neq 0$ and $a^5 = 0$, can we construct x (in terms of a) so that $x^2 = 0$.

6. Show that the set S of all nilpotent elements of a communtative ring R is an ideal, i.e., S is a subring satisfying $ax \in S$ for every $a \in S$ and $x \in R$.

Note Let R_1, R_2 be rings. A function $\phi : R_1 \to R_2$ is a ring homomorphism if

$$\phi(x + y) = \phi(x) + \phi(y) \text{ and } \phi(xy) = \phi(x)\phi(y) \text{ for any } x, y \in R_1.$$

7. Suppose R is a commutative ring with unity and $\text{char}R = p$, where p is a prime. Show that $\phi : R \to R$ defined by $\phi(x) = x^p$ is a ring homomorphism.

   [Hints: You may use the binomial formula $(x+y)^p = \sum_{k=0}^{p} \binom{p}{k} x^k y^{p-k}$ in a commutative ring, and you need to show that p is a factor of $\binom{p}{k}$ if $k = 1, \ldots, p-1$.]

8. Let R_1 and R_2 be rings, and $\phi : R_1 \to R_2$ be a ring homomorphism such that $\phi(R_1) \neq \{0'\}$, where $0'$ is the additive identity of R_2.

   (a) Show that if R_1 has a unity and R_2 has no zero-divisors, then $\phi(1)$ is a unity of $\phi(R_1)$.

   (b) Show that the conclusion in (a) may fail if R_2 has zero-divisors.

   [Hint: (a) Suppose $\phi(x) \neq 0'$. Then $\phi(x) = \phi(1)\phi(x)$ and $\phi(x)\phi(1)$. Then for any $y \in R_2$, consider $\phi(x)(\phi(1)y - y)$ and $(y\phi(1) - y)\phi(x)$. (b) Consider $R_1 = R_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.]