**Math 307 Abstract Algebra     Homework 11                    Sample Texfile**

Five points for each question unless specified otherwise.

1. Find an multiplicative inverse of $2x+1$ in $\mathbb{Z}_4[x]$. Is the inverse unique?

2. (a) Given an example to show that a factor ring of an integral domain may have zero-divisors.

   (b) Give an example to show that a factor ring of a ring with zero-divisors may be an integral domain.

3. Let $R_1$ and $R_2$ be rings, and $\phi : R_1 \to R_2$ be a ring homomorphism.

   (a) Show that if $A$ is an ideal of $R_1$, then $\phi(A)$ is an ideal of $\phi(R_1)$.

   (b) Give an example to show that $\phi(A)$ may not be an ideal of $R_2$.

4. (8 points) Let $A = \langle x^2+x+1\rangle = \{(x^2+x+1)f(x) : f(x) \in \mathbb{Z}_2[x]\} \subseteq \mathbb{Z}_2[x]$.

   (a) Show that $\mathbb{Z}_2[x]/A = \{a+bx+A : a,b \in \mathbb{Z}_2\}$ has 4 elements.

   (b) Show that $(a+bx+A)(c+dx+A) = (ac+bd)+(ad+bc+bd)x+A$.

   (c) For each nonzero element $a+bx+A \in \mathbb{Z}_2[x]/A$, show that there is $c+dx+A \in \mathbb{Z}_2[x]/A$ such that $(a+bx+A)(c+dx+A) = 1+A$, and deduce that $\mathbb{F} = \mathbb{Z}_2[x]/A$ is a field.

   (d) Show that the nonzero elements in $\mathbb{F}$ form a cyclic group under multiplication.

   Hint: (a) $x^2+A = 1+x+A$. (c) Only need to consider the cases when $a+bx+A = 1+A, x+A$, and $1+x+A$. (c) (d) Show that $x+A$ has order 3 under multiplication.

5. (12 points) Let $A = \langle x^2+1\rangle = \{(x^2+1)f(x) : f(x) \in \mathbb{Z}_3[x]\} \subseteq \mathbb{Z}_3[x]$.

   (a) Show that $\mathbb{Z}_3[x]/A = \{a+bx+A : a,b \in \mathbb{Z}_3\}$ has 9 elements.

   (b) Show that $(a+bx+A)(c+dx+A) = (ac+2bd)+(ad+bc)x+A$.

   (c) For each nonzero element $a+bx+A \in \mathbb{Z}_3[x]/A$, show that there is $c+dx+A \in \mathbb{Z}_3[x]/A$ such that $(a+bx+A)(c+dx+A) = 1+A$, and deduce that $\mathbb{F} = \mathbb{Z}_3[x]/A$ is a field.

   (d) Determine the multiplicative inverse of $1+2x+A \in \mathbb{F}$.

   (e) Show that the nonzero elements in $\mathbb{F} = \mathbb{Z}_3[x]/A$ form a cyclic group under multiplication.

   (f) Show that $X = x+A$ is a zero of the polynomial $X^2+1 \in \mathbb{F}[X]$.

   Hint: (c) Just show that every nonzero element has an inverse. (d) Use the hint in (c) to find $f(x)+A$ for a suitable $f(x)$ so that $(1+2x+A)(f(x)+A) = 1+A$. (e) Note that there are 8 nonzero elements of $\mathbb{F}$ that form a group under multiplication. Find $a+bx+A$ such that $(a+bx+A)^4 \neq 1+A$. Then $a+bx+A$ will have order 8.

6. Let $\mathbb{Z}[i] = \{a+ib : a,b \in \mathbb{Z}\}$ be the integral domain $\mathbb{D}$ of Gaussian integers. Let $\mathbb{F} = \{[(a,b)] : a \in \mathbb{D}, b \in \mathbb{D}^*\}$ be the field of quotients of $\mathbb{D}$, and $\mathbb{Q}[i] = \{x+iy : x,y \in \mathbb{Q}\}$.

   (a) Show that $[(a+ib, c+id)] = [(ac+bd)+i(bc-ad), c^2+d^2] \in \mathbb{F}$.

   (b) Show that $\phi : \mathbb{F} \to \mathbb{Q}[i]$ defined by $\phi([(a+ib, c+id)]) = \frac{ac+bd}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$ is an isomorphishm.