Math 307 Abstract Algebra Homework 12 Name: Your name

Five points for each question unless specified otherwise.

1. Show that the ideal $\langle x^2+1\rangle$ is prime in $\mathbb{Z}[x]$, but it is not a maximal ideal.
   [Hint: Consider $\mathbb{Z}[x]/\langle x^2+1\rangle$.]
2. Suppose R is a commutative ring. Show that $R[x]$ and R have the same characteristics.
   Consider two cases. $\mathrm{char}(R) = m$ is finite, $\mathrm{char}(R) = 0$.
3. Let $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{Z}[x]$. If $f(r/s) = 0$, where $r/s \in \mathbb{Q}$ is in its lowest form.
   Show that $r|a_0$ and $s|a_n$.
4. Let $\mathbb{F}$ be a field, and $f(x) \in \mathbb{F}[x]$.
   (a) Show that $(x-1)$ is a factor of $f(x)$ if and only if $a_0 + \cdots + a_n = 0$.
   (b) Show that $(x+1)$ is a factor of $f(x)$ if and only if $a_0 - a_1 + \cdots + (-1)^n a_n = 0$.
5. Suppose $\mathbb{F}$ is a field, and $f(x), g(x) \in \mathbb{F}[x]$ are such that $f(a) = g(a)$ for infinitely many $a \in \mathbb{F}$.
   Show that $f(x) = g(x)$.
6. (10 points) Let $\mathbb{F}$ be a field. For $f(x) = a_0 + \cdots + a_n x^n \in \mathbb{F}[x]$, let
   $$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}.$$
   (a) Prove that if $h(x) = f(x)g(x)$, then $h'(x) = f(x)g'(x) + f'(x)g(x)$.
   (b) Show that a zero a of $f(x) \in \mathbb{F}[x]$ has multiplicity 2, i.e., $(x-a)^2$ is a factor of $f(x)$ if and only if a is a zero of $f(x)$ and a zero of $f'(x)$.

Extra credit problems

7. Find infinitely many polynomials $f(x)$ in $\mathbb{Z}_n[x]$ such that $f(a) = 0$ for all $a \in \mathbb{Z}_n$.
8. If $r \in \mathbb{R}$ such that $r + 1/r \in \mathbb{Z} \setminus \{2, -2\}$, than r is irrational.
9. Let $\mathbb{F}$ be a field. Show that there exist $a, b \in \mathbb{F}$ such that $x^2 + x + 1$ is a factor of
   $x^{43} + ax + b$.
10. (10 points) (Wilson's Theorem) In $\mathbb{Z}_n$, show that $(n-1)! = (n-1)$ if and only if n is a prime.
    [Hint: Consider two cases: n is a prime, n is not a prime.]
    Use this result to deduce the following.
    (a) Find the remainder of 98! divided by 101.
    (b) Show that $(50!)^2 = -1$ in $\mathbb{Z}_{101}$.