MATH 307 Quiz 1 (sample) Time allowed: 20 min Name: Sample Solution.

1. Let S be a relation on $\mathbf{R}$ such that $(a, b) \in S$ if $a - b = 2k$ for some integer k.

Show that S is an equivalent relation, and determine **All** the equivalence classes.

Reflexive: Let $a \in R$. $a - a = 0$ is an even integer. So $(a,a) \in S$.

Symmetric: Let $a, b \in R$, $(a,b) \in S$. Then $a - b = 2k$ So $b - a = 2(-k)$.

Thus $(b,a) \in S$.

Transitive: Let $a, b, c \in R$, $(a,b), (b,c) \in S$. Then $a - b = 2k$, $b - c = 2l$, $k, l \in \mathbb{Z}$.

So $a - c = \cancel{2(k}$ $a - b + b - c = 2(k + l)$. $\therefore$ $(a,c) \in S$.

Let $r \in [0, 2)$. $[r] = \{r + 2k : k \in \mathbb{Z}\} = [r + 2l]$ for any $l \in \mathbb{Z}$.

So, $\{[r] : r \in [0,2)\}$

will be the set of all equivalence classes of S.

2. Let G be a group, and $a \in G$. Show that $(a^{-1})^{-1} = a$.

[Hint: Consider the equation $a^{-1}x = e$.]

Let G be a group. $a \in G$.

$$a^{-1} a = e$$

$$a^{-1} (a^{-1})^{-1} = e.$$

$$\therefore \quad a^{-1} a = a^{-1} (a^{-1})^{-1}.$$

By left cancellation, $a = (a^{-1})^{-1}$.

Re call: $(\mathbb{Q}^*, \cdot) \le (\mathbb{R}^*, \cdot) \le (\mathbb{C}^*, \cdot)$ | $GL_n(\mathbb{Q}^*, \cdot) \le GL_n(\mathbb{R}^*, \cdot) \le GL_n(\mathbb{C}^*, \cdot)$

$(\mathbb{Z}, +) \le (\mathbb{Q}, +) \le (\mathbb{R}, +) \le (\mathbb{C}^*, +)$ | $(M_{m,n}(\mathbb{Z}), +) \le (M_{m,n}(\mathbb{Q}), +) \le \ldots$

Theorems 3.4 – 3.6 Let G be a group, and let $a \in G$. The following are subgroups of G.

(1) The cyclic subgroup generated by a is $\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$.

(2) The centralizer of a is $C(a) = \{g \in G : ga = ag\}$.

(3) The center of G is $Z(G) = \{x \in G : xg = gx \text{ for all } g \in G\}$.

$\boxed{H \neq \emptyset. \text{ Then } H \le G \iff ab^{-1} \in H \text{ whenever } a, b \in H}$

Remarks

- Intersection of subgroups of G is always a subgroup.
- The union of two subgroups of G is a subgroup if and only if one of the subgroups contains the other subgroup.
- The center of a group is Abelian.
- The centralizer of an element may not be Abelian.

Examples: of subgroups.

$G_0 = \{f : f: \mathbb{R} \to \mathbb{R}, \text{ continuous.}\}$

under addition $(f+g) = h$.

$(f+g)(x) = h(x)$

$h(x) = f(x) + g(x)$

e.g. $\sin(x)$, $\cos(x)$

$\sin(x) + \cos(x)$

(G0) ✓

(G1) ✓ $(f+g)+h = \cancel{(f+g)}$

$= f+(g+h) \quad \forall f,g,h \in G$

(G2) $e(x) = 0 \; \forall x$, the zero function satisfies

$\cancel{e(x) + f(x) = f}$ $\quad e+f = f+e = f$.

(G3) For every $f \in G$, define $-f: \mathbb{R} \to \mathbb{R}$ such that

$(-f)(x) = -(f(x))$

Then $f + (-f) = e$.

G_1 = set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$, ~~under~~ is a group under addition

& $G_1 \le G$. Check ① $f, g \in G_1 \Rightarrow f+(-g) \in G_1$

$f - g$

(2) $\varphi(x) = 0 \; \forall x \in \mathbb{R}$

Proof of

Theorem 3.x Let G be a group.

(1) Let $a \in G$. and define

$$\langle a \rangle = \{a^n : n \in \mathbb{Z}\}$$

where $a^n = \underbrace{a * \cdots * a}_{n}$ if n is ~~an~~ a positive integer.

$\cancel{a^n = (a^{-1})^{|n|}}$

$a^{-n} = (a^{-1})^n$ if n is positive integer

$= (a^n)^{-1}$

$a^0 = e$. the identity in G.

Let $H = \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$.

Proof: (1) By definition, $a^0 = e \in H$. $\neq \phi$.

2° ~~Let~~ $x, y \in H$,

Then $x = a^n$, $y = a^m$.

So $xy^{-1} = (a^n)(a^{-m})$

$= a^{n-m} \in H$.

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(2) Let $a \in G$. $C(a) = \{g \in G : ga = ag\} \subseteq G$.

① $e \in G$ satisfies $ea = ae = a$

$\therefore e \in C(a)$.

② $x, y \in C(a)$, $xa = ax$, $ya = ay$

$(xy^{-1})a = xay^{-1} = axy^{-1} = a(xy^{-1})$

$\therefore$

$\therefore$ $xy^{-1} \in C(a)$

Note $ya = ay$.

$\therefore y^{-1}(ya)y^{-1} = y^{-1}(ay)y^{-1}$

$ay^{-1} = y^{-1}a$.

$\mathbb{Z}_{12}$

$\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \mathbb{Z}_{12}$

$\boxed{\langle \bar{2} \rangle = \{\bar{2}, \bar{4}, \bar{6}, \bar{8}, \bar{10}, \bar{0}\}}$

$\langle \bar{3} \rangle = \{\bar{3}, \bar{6}, \bar{9}, \bar{0}\}$

$\langle \bar{4} \rangle = \{\bar{4}, \bar{8}, \bar{0}\}$

$\langle \bar{5} \rangle = \{\bar{1}, \ldots, \bar{0}\} = \mathbb{Z}_{12}$

$\langle \bar{6} \rangle = \{\bar{6}, \bar{0}\}$

$\langle \bar{7} \rangle = \mathbb{Z}_{12}$

$\langle \bar{8} \rangle = \{\bar{4}, \bar{8}, \bar{0}\}$

$\langle \bar{9} \rangle$

$\langle \bar{10} \rangle$

$\langle \bar{11} \rangle$

(3) Let G be a group. $Z(G) = \{x \in G : xa = ax \ \forall a \in G\}$

$$= \bigcap_{a \in G} C(a)$$

(i) $ea = ae \ \forall a \in G \quad \therefore \ e \in Z(G)$

(2) $x, y \in Z(G)$. $\quad xa = ax \ \forall a \in G$

$\left\{ \begin{array}{l} ya = ay \ \forall a \in G \\ \Rightarrow y^{-1}a = ay^{-1} \ \forall a \in G. \end{array} \right.$

$\therefore \ (xy^{-1}a)$

$= xay^{-1} = axy^{-1}$

$\therefore \ (xy^{-1})a = a(xy^{-1}) \ \forall a \in G.$

More results on subgroups

Theorem 3.3 Let G be a group and H be a **non-empty finite** subset of G. Then H is a subgroup of G if and only if $ab \in H$ whenever $a, b \in H$.

Proof: Given

① $H \neq \phi$

② $a, b \in H \Rightarrow ab \in H$

TS (1) It remain to prove that

"$a \in H \Rightarrow a^{-1} \in H$"

Important Idea

Let $a \in H$.

Consider

$a, a^2, a^3, \cdots, a^n, \cdots \in H$

∵ These elements in H cannot be all different.

So there is a $p < q$ such that $a^p = a^q$

In G, $a^{q-p} = e$

Case 1° $q - p = 1$ $a^1 = e$, $a^{-1} = e \in H$

Case 2° $q - p > 1$. Then $a \cdot a^{q-p-1} = e$. ∴ $a^{-1} = a^{q-p-1}$

∴ $a^{-1} = \boxed{a^{q-p-1}} \in H$

Find example that

G infinite

H infinite

$a, b \in H \Rightarrow ab \in H$

$H \not\leq G$

$(\mathbb{Z}, +)$

$(\mathbb{N}, +)$