Group of permutations S_A

S_n = set of bijections on $\{1, \dots, n\}$

Symmetric group of degree n.

Representations: $\sigma = \begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$

$\sigma(1) = i_1, \ \sigma(2) = i_2 \ \dots \ \sigma(n) = i_n$

~~Let m~~

$\sigma^m = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{m}$ if ~~map~~ m is a positive integer

$\sigma^0 = I$ identity map

$\sigma^{-m} = (\sigma^m)^{-1} = (\sigma^{-1})^m$ $\quad m$ a positive integer

Disjoint Cycle representation

$\sigma = C_1 \circ \cdots \circ C_k$ disjoint cycles.

$C_i = (i_1, \dots, i_k)$ corresponds to the bijection f such that

$f(i_1) = i_2, \ f(i_2) = i_3, \ \dots, \ f(i_{k-1}) = i_k, \ f(i_k) = i_1.$

and $f(j) = j$ otherwise.

$\sigma = \tau_1 \cdots \tau_k$. is a product of transpositions.

$\tau_i = (i_1, i_2)$.

Every σ is either an even or an odd permutation.

The identity $I = (12)(12)(23)(23)$

Theorem:

Let A_n be the set of even permutations in S_n.

① A_n is a subgroup of S_n, called the alternating group.

② $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Proof: ①

1° $A_n \neq \phi$ because the identity permutation $\tau = (1\,2)(1,2)$ is an even permutation.

2° Suppose $\sigma_1, \sigma_2 \in A_n$, i.e.,
$\sigma_1 = \tau_1 \cdots \tau_{2l}$, $\sigma_2 = \hat{\tau}_1 \cdots \hat{\tau}_{2k}$, $\tau_1, \cdots, \tau_{2l}, \hat{\tau}_1, \cdots \hat{\tau}_{2k}$ are transpositions.
So $\sigma_1\sigma_2 = \tau_1 \cdots \tau_{2l}\hat{\tau}_1 \cdots \hat{\tau}_{2k}$ is a product of even number of transpositions.
$\therefore \sigma_1\sigma_2 \in A_n$

3° Suppose $\sigma \in A_n$, i.e., $\sigma = \tau_1 \cdots \tau_{2l}$, $\tau_1, \cdots, \tau_{2l}$ is a transposition
$\sigma^{-1} = \tau_{2l}^{-1} \cdots \tau_1^{-1} = \tau_{2l} \cdots \tau_1$ is the product of an even number of transpositions.
$\therefore \sigma^{-1} \in A_n$

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② Consider $f: A_n \to (S_n - A_n)$ by
$f(\sigma) = (1,2)\sigma$

Claim: f is a bijection.

Well-defined: Suppose $\sigma \in A_n$, i.e., $\sigma = \tau_1 \cdots \tau_{2l}$, $\tau_1, \cdots, \tau_{2l}$ are transpositions.
Then $f(\sigma) = \underline{(1\,2)}\,\tau_1 \cdots \tau_{2l}$ is an odd permutation
$\therefore f(\sigma) \in S_n - A_n$

One-one: $f(\sigma_1) = f(\sigma_2) \Rightarrow (1,2)\sigma_1 = (1,2)\sigma_2$
$\Rightarrow \sigma_1 = \sigma_2$ by cancellation

Onto: If $\hat{\sigma} \in S_n - A_n$, let $\sigma = (1,2)\hat{\sigma} \in A_n$ ∵ $\hat{\sigma}$ is odd

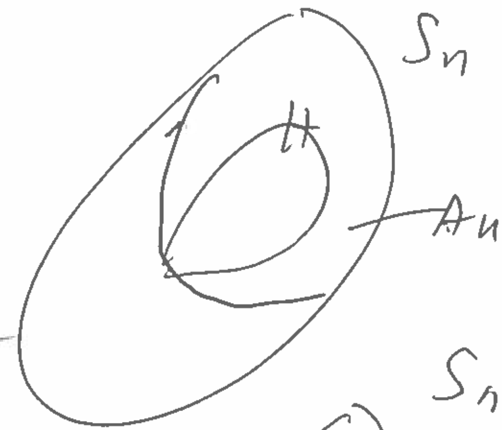
So that $f(\sigma) = (1\,2)(1\,2)\hat{\sigma} = \hat{\sigma}$.

$\therefore \ |A_n| = |S_n - A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$

Hint of Homework: $\quad H \leq S_n$

Then ① $H \leq A_n$

② $|H \cap A_n| = \frac{|H|}{2}$



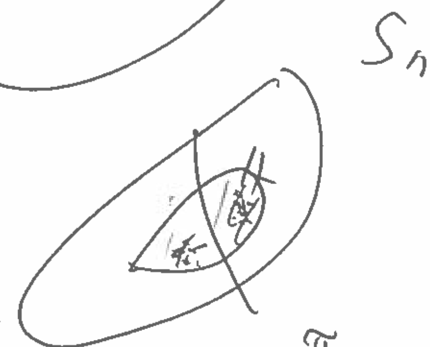
Assume $\quad H \leq S_n$.

If $\ H \leq A_n$ then ① holds.

If $\ H \nleq A_n$, then

$H \cap A_n \neq \emptyset \quad (\because e \in H \cap A_n)$

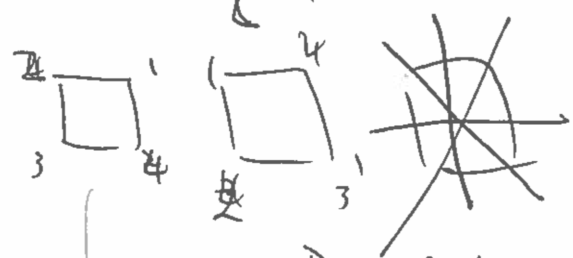
& $H \cap (S_n - A_n) \neq \emptyset$, i.e., $\exists$ an odd permutation $\pi \in H$.



Define $\ f : H \cap A_n \to (H - A_n)$

by $\quad f(\sigma) = \pi\sigma$.

$|\langle R_{90} \rangle| = 4$



Subgroups of permutation groups and applications

(1) Dihedral groups ;

(2) Rotations of tetrahedron.

(3) Moves in Rubik's cube.

(4) Construct check digit using D_5; see pp. 115-116.

All configuration of the Rubik's cube can be regarded as a subgroup of S_{54}

$D_4 =$

D_n Symmetry group of n-sided regular polygons with 2n elements

$\in S_4$