MATH 307 Quiz 3 Name: Sample Solution.

1. Show that the order of $(1,1) \in \mathbf{Z}_m \oplus \mathbf{Z}_n$ is $\operatorname{lcm}(m,n)$.

$(1,1)^k = (0,0)$ iff $(\underbrace{1+\cdots+1}_{k}, \underbrace{1+\cdots+1}_{k}) = (0,0)$ in $\mathbb{Z}_m \oplus \mathbb{Z}_n$.

So k is a multiple of m & k is a multiple of n.

To find the smallest k satisfying the above condition,

we have $k = \operatorname{lcm}(m,n)$.

So the order of $(1,1)$ is $\operatorname{lcm}(m,n)$.

2. Suppose $G = H \oplus K$. Show that $H \oplus \{e\} = \{(x,e) : x \in H\}$ is a normal subgroup of G.
[Hint: Prove that $(h,k)(x,e)(h,k)^{-1} \in H \oplus \{e\}$ for every $(h,k) \in G$.]

(e_H, e_K) $= H \oplus K$.

To prove $H \oplus \{e\} = \{(x,e) : x \in H\}$ is a normal subgroup of G.

Let $(h,k) \in G$. Then for any $(x,e) \in H \oplus \{e\}$,

$$(h,k)(x,e)(h,k)^{-1} = (hxh^{-1}, kek^{-1}) = (hxh^{-1}, e) \in H \oplus \{e\}$$

because $hxh^{-1} \in H$.

$\therefore$ $H \oplus \{e\}$ is a normal subgroup of G.

One may check subgroup property of $H \oplus \{e\}$ for Completeness: ① $(e_H, e) \in H \oplus \{e\}$ is non-empty

② If $(h_1, e), (h_2, e) \in H \oplus \{e\}$

then $(h_1,e)(h_2,e)^{-1} = (h_1 h_2^{-1}, e) \in H \oplus \{e\}$

because $h_1 h_2^{-1} \in H$.

$\mathbb{Z}_{p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots p_k^{r_k}} \cong \mathbb{Z}_{p_1^{r_1}} \times \cdots \times \mathbb{Z}_{p_k^{r_k}}$ $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_5$ $\mathbb{Z}_{36} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_9$

$H, K \le G$ $\boxed{H \oplus K \cong G}$ $\mathbb{Z}_{36} \not\cong \mathbb{Z}_6 \oplus \mathbb{Z}_6$

$H \oplus \{e\} \triangleleft H \oplus K \cong G$

$\{e\} \oplus K \triangleleft H \oplus K \cong G$

Additional results

Definition If H, K are normal subgroups of G such that $\boxed{H \cap K = \{e\}}$ and $HK = G$, then G is the internal product of H and K. For more than 2 groups, we need $G = H_1 \cdots H_k$ and $(H_1 \cdots H_j) \cap H_{j+1} = \{e\}$.

Theorem 9.6 Internal direct product $G_1 \cdots G_k$, sometimes denoted by $G_1 \times \cdots \times G_k$, is isomorphic to $G_1 \oplus \cdots \oplus G_k$.

Proof. Consider $\phi : \boxed{H \oplus K} \to \boxed{H \times K}$ by $\phi(h,k) = hk$. (h,k) (e,e) $e \in G$

$HK = G$

ϕ is one-one: If $\phi(h_1,k_1) = \phi(h_2,k_2)$ then $h_1k_1 = h_2k_2 \in G$

Thus, $h_1^{-1}h_2 = k_1k_2^{-1} \in H \cap K = \{e\}$, and $\therefore h_1^{-1}h_2 = e \Rightarrow h_1 = h_2$ Also $k_1k_2^{-1} = e \Rightarrow k_1 = k_2$

ϕ is onto: If $g \in G$, then $g = hk$ for some $h \in H, k \in K$. So, $(h,k) \in H \oplus K$ s.t. $\phi(h,k) = hk$.

Finally to show that $\phi((h_1,k_1)(h_2,k_2)) = \phi(h_1h_2, k_1k_2) = h_1h_2k_1k_2$ equals $\phi(h_1,k_1)\phi(h_2,k_2) = h_1k_1h_2k_2$, we need to show that $h_2k_1 = k_1h_2$.

This is true because $k_1^{-1}(k_1h_2)h_1^{-1}h_2 \in H$

$$\boxed{k_1^{-1}}\boxed{h_2k_1}h_1^{-1}h_2 = \tilde{h}h_2 = \boxed{k_1^{-1}\tilde{k}} \in H \cap K = \{e\}.$$

$k_1^{-1}(h_2k_1)h_1^{-1}h_2$ $(h_2k_1) = (k_1h_2)$ $k_1^{-1}k_1h_2h_1^{-1}h_2$

We can extend the proof to $G = H_1 \otimes \cdots \otimes H_k$. □

Remark Note that if $G \cong H_1 \times \cdots \times H_k$, then $G \cong H_1 \oplus \cdots \oplus H_k$. However, the converse is not true. For example, if $G = \mathbf{Z} \oplus \mathbf{Z}$, then $H_1 = \mathbf{Z} \oplus \{0\}$ and $H_2 = \{0\} \oplus 2\mathbf{Z}$ are normal subgroup of G such that $H_1 \oplus H_2 \cong \mathbf{Z} \oplus \mathbf{Z} = G$, but $G \not\cong H_1 \times H_2$.

Theorem 9.7 A group of order p^2 for a prime p is isomorphic to $\mathbf{Z}_{p^2}$ or $\mathbf{Z}_p \oplus \mathbf{Z}_p$. In either case, G is Abelian.

Proof. Suppose there is $a \in G$ of order p^2. Then we are done.

Assume that every nonidentity element $a \in G$ has p.

We can find $a \in G$ such that $H = \langle a \rangle$ has order p.

We show that H is normal. If not, there is $b \in G - H$ such that $bHb^{-1} \not\subseteq H$. So, $bab^{-1} \notin H$; otherwise $\boxed{\langle a \rangle = \langle bab^{-1} \rangle = \tilde{H}.}$ Now, $b^{-1} \in \cup_{j=0}^{p-1} a^j \tilde{H} = G$. So, $b^{-1} = a^i(bab^{-1})^j$ for some i, j, and hence $e = a^iba^j$ implying that $b = a^{-i-j}$, which is a contradiction. $= a^iba^jb^{-1}$

Now, let $b \in G - H$ and $\tilde{H} = bHb^{-1}$. One see that $H, \tilde{H}$ are normal subgroup of G such that $H \cap \tilde{H} = \{e\}$. Hence, $G = H \times \tilde{H} \cong \mathbf{Z}_p \otimes \mathbf{Z}_p$.

Note that $bHb^{-1} = \{ba^jb^{-1} : 0 \le j \le p-1\}$ is a subgroup. If $ba^jb^{-1} \notin H$ □

~~$h_2k_1 = k_1h_2$~~ ??

iff

$h_2^{-1}k_1^{-1}(h_2k_1)$

$= h_2^{-1}(k_1^{-1}h_2k_1) = h_1^{-1}\hat{h} \in H$

and $(h_2^{-1}k_1^{-1}h_2)k_1 = \hat{k}k_1 \in K$