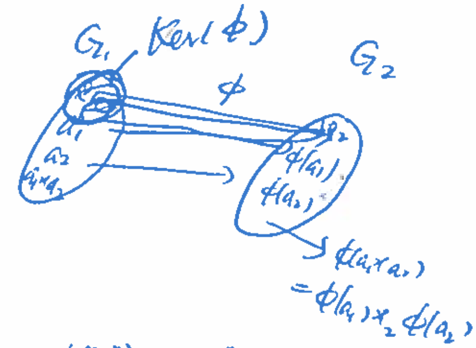
Chapter 10 Group Homomorphisms

Definition Let $(G_1, *_1), (G_2, *_2)$ be groups. Then a function $\phi : G_1 \to G_2$ is a group homomorphism if

$$\phi(a *_1 b) = \phi(a) *_2 \phi(b) \quad \text{for all } a, b \in G_1.$$



The **kernel** of ϕ is the set $Ker(\phi) = \{a \in G_1 : \phi(a) = e_2\}$.

Example $\phi : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$, $\phi(x) = kx$.

Example $\phi : \mathbf{Z}_{12} \to \mathbf{Z}_{30}$, $\phi(x) = kx$. $k = 1, 2, 3, 4$

Example $\phi : S_n \to \{1, -1\}$, $\phi(\sigma) = 1$ if σ is even; else $\phi(\sigma) = -1$.

$\phi([x]) = [x]$ the identity map, $\phi([x+y]) = \phi([x]) + \phi([y]) = [x]+[y]$
$\qquad = [x]+[y]$

$\phi([x]) = \underline{2[x]} = [2x]$. $\qquad \phi([0]) = [0]$

$[x]$	$[0]$, $[1]$, $[2]$, $[3]$, …	$[11]$
$\phi([x])$	$[0]$, $[2]$, $[4]$, $[6]$, …	$[10]$

1° well-defined:

If $[x] = [y]$, i.e., $x - y = 12k$

then $\phi([x]) = [2x]$

& $\phi([y]) = [2y]$ s.t. $2x - 2y = 2(x-y) = 12(2k)$.

$\therefore \phi([x]) = [2x] = [2y] = \phi([y])$

Not onto: From the list, ~~$\phi([x]) \neq$~~ $\phi([x]) \neq [1]$ for any $[x] \in \mathbb{Z}_{12}$.

~~Because $[\frac{1}{2}] \notin \mathbb{Z}_{12}$, $\therefore$, $\nexists$ not $[x] \in \mathbb{Z}_{12}$ satisfies $\phi([x]) = [1]$,~~

Suppose $[x] \in \mathbb{Z}_{12}$ such that $\Phi([x]) = [2x] = [1]$.

Then $2x - 1 = 12k$ for some $k \in \mathbb{Z}$, i.e., $2x = 12k + 1$, a contradiction.

Not one-one: $\Phi([0]) = [0]$

$\Phi([6]) = [12] = [0]$.

$Ker(\phi) = \{[x] \in \mathbb{Z}_{12} : \Phi([x]) = [2x] = [0]\}$

$= \{[x] \in \mathbb{Z}_{12} : 2x - 0 = 12k\} = \{[0], [6]\}$.

Example $\phi([x]) = 3[x] = [3x]$

Example $\phi[x] = k[x] = [kx]$.

Well-defined $[x]=[y]$ i.e., $x-y = 12q$, $q \in \mathbb{Z}$

$\therefore$ $\phi([x]) - \phi([y])$

$= [kx] - [ky]$

$= [k(x-y)]$

$= [k \cdot 12 q] = [0]$

Rmk: ① Each choice of $k = 0, 1, \cdots, 11$, gives rise to a different homo.

② These 12 homomorphisms are all homo. from $\mathbb{Z}_{12} \to \mathbb{Z}_{12}$

Reason: Suppose $\phi: \mathbb{Z}_{12} \to \mathbb{Z}_{12}$ is a homo.

& $\phi([1]) = [k]$, $k = 0, \cdots, 11$.

Then $\phi([m]) = \phi([1] + \cdots + [1])$

$$= \underbrace{\phi([1]) + \cdots + \phi([1])}_{m}$$

$= [mk]$.

③ $\phi([x]) = [kx]$ is an isom.

$\Leftrightarrow$ $\gcd(k, n) = 1$.

Because if $\gcd(k, n) = 1$

$[km] = [0]$. can only happen

when $km = 12q$.

Define

$\phi: \mathbb{Z}_{12} \to \underline{\mathbb{Z}_{30}}$ by $\phi([1]) = [k]$.

So that $\phi([m]) = [km]$

is well-defined provided :

$[12]-[0] = 12 \cdot q$

$[x] = [y]$ in $\mathbb{Z}_{12}$, i.e., $x - y = 12q$

Then $\phi([x]) = [kx]$

& $\phi([y]) = [ky]$ satisfy

$kx - ky = k(x-y) = \boxed{12k} \cdot \cancel{(x-y)}\, q$

is multiple of $\underline{30}$.

In particular, $12k$ is a multiple of 30

$12k = 30m$

$2k = 5m$

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$\phi[1] = [0]_{30}$

$\phi[1] = [5]_{30}$

$\phi[1] = [10]_{30}$

$\phi[1] = [15]_{30}$

$\phi[1] = [20]_{30}$

$\phi[1] = [25]_{30}$

$5 \mid k$. $\phi([x]) = [kx]$

$Ker(\phi) = \{[r]_{12} : \phi([r]_{12}) = [0]_{30}\}$

$= \{[r]_{12} : [kr]_{30} = [0]\}$.

Example: $\phi: \mathbb{Z}_m \oplus \mathbb{Z}_n \to G$.

$\phi([1],[0]) = g_1$, $\phi([0],[1]) = g_2$ $\to G$

Then ~~$\phi([r], s$~~ $\phi([r],[s]) = \phi([r],[0]) * \phi([0],[s])$

$= \underbrace{g_1 * g_1 * \cdots g_1}_{r} * \underbrace{g_2 * \cdots * g_2}_{s}$

Example $\phi: S_n \to \{1,-1\}$, $\phi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{otherwise.} \end{cases}$

Then ϕ is welldefined

ϕ is homo. because of the following:

Let $\sigma_1, \sigma_2 \in S_n$.

1° $\sigma_1, \sigma_2 \in A_n$ Then $\phi(\sigma_1) = \phi(\sigma_2) = 1$

$\phi(\sigma_1\sigma_2) = 1 = \phi(\sigma_1)\phi(\sigma_2)$

2. $\sigma_1 \in A_n$, $\sigma_2 \notin A_n$

3. $\sigma_1 \notin A_n$ $\sigma_2 \in A_n$

$\left.\begin{matrix} \phi(\sigma_1\sigma_2) = -1 = \phi(\sigma_1)\phi(\sigma_2) \\ \phi(\sigma_1\sigma_2) = -1 = \phi(\sigma_1)\phi(\sigma_2) \end{matrix}\right.$

4. $\sigma_1, \sigma_2 \notin A_n$. Then $\phi(\sigma_1) = -1 = \phi(\sigma_2)$

$\phi(\sigma_1\sigma_2) = 1 = \phi(\sigma_1)\phi(\sigma_2)$

$\ker(\phi) = A_n$

Theorem 10.1-2 Suppose $\phi : G_1 \to G_2$ is a group homomorphism.

$x \in G$

$x^k = e$

1. $\phi(e_1) = e_2$.
2. $\phi(g^n) = \phi(g)^n$ for all $n \in \mathbf{Z}$.
3. If $m = |\phi(g)|$ and $n = |g|$, then m divides n.
4. $Ker(\phi)$ is a subgroup of G_1; $\phi(a) = \phi(b)$ if and only if $aKer(\phi) = bKer(\phi)$.
5. If $|Ker(\phi)| = n$, then f is an n to 1 function, and $n|\phi(H)| = |H|$. In particular, ϕ is 1-1 if and only if $|Ker(\phi)| = 1$.

Cor

6. If H is a (cyclic, Abelian) subgroup of G_1, then $\phi(H) = \{\phi(a) : a \in H\}$ is a (cyclic, Abelian) subgroup of G_2; if H is a normal subgroup in G_1 then $\phi(H)$ is a normal subgroup in $\phi(G_1)$.
7. If K is a (normal) subgroup of G_2, then $\phi^{-1}(K) = \{a \in G_1 : \phi(a) \in K\}$ is a (normal) subgroup of G_1. In particular, $Ker(\phi)$ is normal in G.

Remark Consider $\phi : \mathbf{Z}_2 \to S_3$ such that $\phi(1) = (1,2)$. Then $H = \mathbf{Z}_2$ is a normal subgroup in $\mathbf{Z}_2$, but $\phi(\mathbf{Z}_2) = \{\varepsilon, (1,2)\}$ is not a normal subgroup in S_3.

Proof

~~Let $\phi(e_1) = y \in G_2$~~ Note:

(1) $e_2 \phi(e_1) = \phi(e_1) = \phi(e_1 * e_1) = \phi(e_1) * \phi(e_1)$

$\therefore e_2 = \phi(e_1)$ by Cancellation in G_2

(2) $\phi(g^n) = \phi(\underbrace{g * \cdots * g}_{n}) = \phi(g)^n \quad \forall n \in \mathbb{N}$.

$\phi(g^0) = \phi(e_1) = e_2 = \phi(g)^0$

To study ~~with~~ $n \in \mathbb{N}$. $\phi(g^{-n})$

Consider $\phi(g^n g^{-n}) = \phi(e_1) = e_2$

$\| $

$\phi(g^n)\phi(g^{-n})$

$\|$

$\phi(g)^n \phi(g^{-n})$ $\quad \therefore \phi(g^{-n}) = \phi(g)^{-n}$

$y^n \cdot \phi(g^{-n}) = e_2$

$\phi(g^{-n}) = y^{-n}$

Proof: (3) Let $m = |\phi(g)|$, $n = |g|$.

$$g^n = e_1 \qquad \phi(g)^m = e_2.$$

$$\phi(g)^n = \phi(g^n) = \phi(e_1) = e_2.$$

$\therefore |\phi(g)|$ divides n.

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(4) $\mathrm{Ker}(\phi) = \{a \in G_1 : \phi(a) = e_2\}$

① $e_1 \in \mathrm{Ker}(\phi)$ $\because \phi(e_1) = e_2$. $\therefore \mathrm{Ker}(\phi) \neq \phi$

② $a, b \in \mathrm{Ker}(\phi)$, i.e., $\phi(a) = e_2 = \phi(b)$

$\therefore \phi(ab) = \phi(a)\phi(b) = e_2 e_2 = e_2$ $\therefore ab \in \mathrm{Ker}(\phi)$

③ If $\phi(a) = e_2$, then $\phi(a^{-1}) = \phi(a)^{-1} = e_2^{-1} = e_2$.

$\therefore a^{-1} \in \mathrm{Ker}(\phi)$

Combining, $\mathrm{Ker}(\phi)$ is a subgroup in G_1

Actually $\mathrm{Ker}(\phi)$ is a normal subgroup.

$H \trianglelefteq G$
$\Leftrightarrow aHa^{-1} \leq H$
$\Leftrightarrow \forall a \in G, x \in H$, $axa^{-1} \in H$

Proof: Let $a \in G$, $x \in \mathrm{Ker}(\phi)$

Then $\phi(axa^{-1}) = \phi(a)\phi(x)\phi(a^{-1})$ e_2

$= \phi(a)\, e_2\, \phi(a)^{-1} = e_2$.

$\therefore axa^{-1} \in \mathrm{Ker}(\phi)$.

(4) $\phi(a) = \phi(b) \Leftrightarrow \phi(a)^{-1}\phi(b) = e_2$

$\Leftrightarrow \phi(a^{-1}b) = e_2 \Leftrightarrow a^{-1}b \in \mathrm{Ker}(\phi)$

$\Leftrightarrow a\,\mathrm{Ker}(\phi) = b\,\mathrm{Ker}(\phi)$

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(5) $\qquad\qquad\qquad\qquad \{\phi(x) : x \in H\}$

(6) Let $H \le G_1$. Consider $\phi(H) \subseteq G_2$.

① ~~$\phi(e_1) = e_2$~~ . $e_1 \in H$, $\phi(e_1) = e_2 \in \phi(H) \neq \phi$.

② ~~$a, b \in H, \Rightarrow \phi\, ab^{-1} \in H$~~

~~$\Rightarrow \phi(ab^{-1})$~~

Let $y_1, y_2 \in \phi(H)$. Then $\exists\, x_1, x_2 \in H$ s.t. $\phi(x_1) = y_1$, $\phi(x_2) = y_2$. So $x_1 x_2^{-1} \in H$. So $\phi(x_1 x_2^{-1}) = \phi(x_1)\phi(x_2)^{-1}$

$= y_1 y_2^{-1} \in \phi(H)$.

(b) If $H \trianglelefteq G_1$, then $\phi(H) \trianglelefteq \underline{\underline{\phi(G_1)}}$ not G_2

<u>Proof</u>: $\phi(H) \subseteq \phi(G_1)$

and $\phi(H)$ is a subgroup

$\therefore \phi(H) \le \phi(G_1)$.

To prove normality: pick $a \in \phi(G_1)$, $x \in \phi(H)$.

<u>Have to show</u>: $a x a^{-1} \in \phi(H)$

$a \in \phi(G_1) \Rightarrow \exists\, g \in G_1$ s.t. $\phi(g) = a$

$x \in \phi(H) \Rightarrow \exists\, h \in H$, s.t. $\phi(h) = x$

$\therefore$ ~~$\phi()$~~ $\underline{ghg^{-1}} \in H \trianglelefteq G_1$

So $\phi(ghg^{-1}) \in \phi(H)$

$\phi(g)\phi(h)\phi(g)^{-1} = a x a^{-1}$.