Chapter 10 Group Homomorphisms

Definition Let $(G_1, *_1), (G_2, *_2)$ be groups. Then a function $\phi : G_1 \to G_2$ is a group homomorphism if

$$\phi(a *_1 b) = \phi(a) *_2 \phi(b) \quad \text{for all } a, b \in G_1.$$

The **kernel** of ϕ is the set $Ker(\phi) = \{a \in G_1 : \phi(a) = e_2\}$.

Example $\phi : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$, $\phi(x) = kx$.

Example $\phi : \mathbf{Z}_{12} \to \mathbf{Z}_{30}$, $\phi(x) = kx$.

Example $\phi : S_n \to \{1, -1\}$, $\phi(\sigma) = 1$ if σ is even; else $\phi(\sigma) = -1$.

Theorem 10.1-2 Suppose $\phi : G_1 \to G_2$ is a group homomorphism.

1. $\phi(e_1) = e_2$.
2. $\phi(g^n) = \phi(g)^n$ for all $n \in \mathbf{Z}$.
3. If $m = |\phi(g)|$ and $n = |g|$, then m divides n.
4. $Ker(\phi)$ is a subgroup of G_1; $\phi(a) = \phi(b)$ if and only if $aKer(\phi) = bKer(\phi)$.
5. If $|Ker(\phi)| = n$, then f is an n to 1 function, and $n|\phi(H)| = |H|$. In particular, ϕ is 1-1 if and only if $|Ker(\phi)| = 1$.
6. If H is a (cyclic, Abelian) subgroup of G_1, then $\phi(H) = \{\phi(a) : a \in H\}$ is a (cyclic, Abelian) subgroup of G_2; if H is a normal subgroup in G_1 then $\phi(H)$ is a normal subgroup in $\phi(G_1)$.
7. If K is a (normal) subgroup of G_2, then $\phi^{-1}(K) = \{a \in G_1 : \phi(a) \in K\}$ is a (normal) subgroup of G_1. In particular, $Ker(\phi)$ is normal in G.

Remark Consider $\phi : \mathbf{Z}_2 \to S_3$ such that $\phi(1) = (1,2)$. Then $H = \mathbf{Z}_2$ is a normal subgroup in $\mathbf{Z}_2$, but $\phi(\mathbf{Z}_2) = \{\varepsilon, (1,2)\}$ is not a normal subgroup in S_3.

Theorem 10.3 If $\phi : G_1 \to G_2$, then the map $\Phi : G_1/Ker(\phi) \to \phi(G_1)$ defined by $\Phi(gKer(\phi)) = \phi(g)$ is an isomorphsim from $G_1/Ker(\phi)$ to $\phi(G_1)$.

Proof. Let $K = Ker(\phi)$.

(1) Φ is well-defined because $\Phi(aK) = \Phi(bK)$ implies ...

(2) Φ is 1-1 because ...

(3) Φ is onto because ...

(4) $\Phi(aKbK) = ... = \phi(aK)\phi(bK)$.

Theorem 10.4 A subgroup N is normal in G if and only if it is $N = Ker(\phi)$ for some group homomorphism from G to $\tilde{G}$.

Proof. If $N = Ker(\phi)$ then it is normal. If N is normal then $\phi : G \to G/N$ by $\phi(g) = gN$ is a homomorphism and $Ker(\phi) = N$.

Chapter 11 Fundamental Theorem of Finitely Generated Abelian Group

Theorem 11.1 Every finitely generated Abelian group is isomorphic to a direct product of $\mathbf{Z}_{m_1} \oplus \cdots \oplus \mathbf{Z}_{m_k} \oplus \mathbf{Z}^{\beta}$, where $m_j = p_j^{n_j}$ for some prime number p_j and positive integer n_j for each j, and a nonnegative integer β known as the Betti number.

Corollary If G is a finite Abelian group, and m divides $|G|$, then G has a subgroup of order m.

(But, there may not be an element of order m.)

Isomorphic classes of subgroups

Suppose $|G| = 8, 10, p^2, pq$, etc.