MATH 307 Quiz 1 (sample) Time allowed: 20 min Name: ____________________

1. Let R be a relation on $\mathbf{Z}$ such that $(a, b) \in R$ if $a - b$ is even. Show that R is an equivalent relation, and determine **All** the equivalence classes.

Solution. Reflexive. If $a \in \mathbf{Z}$ then $a - a = 0$ is even. So, $(a, a) \in R$.

Symmetric. If $a, b \in \mathbf{Z}$ and $(a, b) \in R$, then $a - b = 2k$ is even so that $b - a = -2k$ is even, i.e., $(b, a) \in R$.

Transitive. If $a, b, c \in \mathbf{Z}$ such that $(a, b), (b, c) \in R$, then $a - b = 2k, b - c = 2\ell$ are even so that $a - c = a - b + b - c = 2(k + \ell)$ is even. So, $(a, b) \in R$.

Note that $[0] = \{2k : k \in \mathbf{Z}\}$ and $[1] = \{2k + 1 : k \in \mathbf{Z}\}$ are all the equivalence classes.

2. Suppose G is a group, and $a \in G$. For any positive integer n show that $(a^n)^{-1} = (a^{-1})^n$, where $x^n = x * \cdots * x$ for n times if $x \in G$.

Solution. Let $e \in G$ be the identity. We prove the statement by induction on n.
If $n = 1$, then $a^{-1} = a^{-1}$ trivially holds.
Suppose $(a^k)^{-1} = (a^{-1})^k$ for $k \geqslant 1$. Then

$$(a^{k+1})^{-1} = (a^k a)^{-1} = a^{-1}(a^k)^{-1} = a^{-1}(a^{-1})^k = (a^{-1})^{k+1},$$

where the third equality follows from the induction assumption.

By the principle of MI, the result follows.